

Chapter 1

Introduction to Categories and Categorical Logic

Samson Abramsky and Nikos Tzevelekos

Oxford University Computing Laboratory
Wolfson Building, Parks Road, Oxford OX1 3QD, U.K.

Preface

The aim of these notes is to provide a succinct, accessible introduction to some of the basic ideas of category theory and categorical logic. The notes are based on a lecture course given at Oxford over the past few years. They contain numerous exercises, and hopefully will prove useful for self-study by those seeking a first introduction to the subject, with fairly minimal prerequisites. The coverage is by no means comprehensive, but should provide a good basis for further study; a guide to further reading is included.

The main prerequisite is a basic familiarity with the elements of discrete mathematics: sets, relations and functions. An Appendix contains a summary of what we will need, and it may be useful to review this first. In addition, some prior exposure to abstract algebra — vector spaces and linear maps, or groups and group homomorphisms — would be helpful.

Contents

1	Introduction to Categories and Categorical Logic	1
	Samson Abramsky and Nikos Tzevelekos	
1.1	Introduction	7
1.1.1	From Elements To Arrows	8
1.1.2	Categories Defined	9
1.1.3	Diagrams in Categories	10
1.1.4	Examples	12
1.1.5	First Notions	14
1.1.6	Exercises	17
1.2	Some Basic Constructions	17
1.2.1	Initial and Terminal Objects	17
1.2.2	Products and Coproducts	19
1.2.3	Pullbacks and Equalisers	24
1.2.4	Limits and Colimits	27
1.2.5	Exercises	27
1.3	Functors	27
1.3.1	Basics	28
1.3.2	Further Examples	30
1.3.3	Contravariance	32
1.3.4	Properties of Functors	33
1.3.5	Exercises	35
1.4	Natural Transformations	35
1.4.1	Basics	35
1.4.2	Further Examples	38
1.4.3	Functor Categories	40
1.4.4	Exercises	41
1.5	Universality and Adjoints	42
1.5.1	Adjunctions for Posets	43
1.5.2	Universal Arrows and Adjoints	46
1.5.3	Limits and Colimits	52
1.5.4	Exponentials	53

1.5.5	Exercises	55
1.6	The Curry-Howard Correspondence.....	56
1.6.1	Logic	57
1.6.2	Computation	58
1.6.3	Simply-Typed λ -calculus	62
1.6.4	Categories	66
1.6.5	Categorical Semantics of Simply-Typed λ -calculus..	67
1.6.6	Completeness?.....	70
1.6.7	Exercises	73
1.7	Linearity	73
1.7.1	Gentzen Sequent Calculus	74
1.7.2	Linear Logic	75
1.7.3	Linear Logic in Monoidal Categories.....	78
1.7.4	Beyond the Multiplicatives	82
1.7.5	Exercises	84
1.8	Monads and Comonads	85
1.8.1	Basics	86
1.8.2	(Co)Monads of an Adjunction	88
1.8.3	The Kleisli Construction	89
1.8.4	Modelling of Linear Exponentials	92
1.8.5	Exercises	97
1.9	Review of Sets, Functions and Relations.....	98
1.10	Guide to Further Reading	100
	References	101

List of Tables

1.1	Examples of Limits and Colimits.	27
1.2	The Curry-Howard Correspondence.	56
1.3	Natural Deduction System for \wedge, \supset	57
1.4	Examples of λ -terms.	59
1.5	Simply-Typed λ -calculus for \times, \rightarrow	63
1.6	Correspondence between Logic and Computation.	65
1.7	Correspondence between Logic and Categories.	66
1.8	Structural Rules for Logic.	74
1.9	Gentzen Sequent Calculus for \wedge, \supset	75
1.10	Rules for \otimes, \multimap -Linear Logic.	76
1.11	Linear λ -calculus for \otimes, \multimap	77
1.12	Categorical Translation of \otimes, \multimap -Linear Logic.	82

1.1 Introduction

Why study categories — what are they good for? We can offer a range of answers for readers coming from different backgrounds:

- For **mathematicians**: category theory organises your previous mathematical experience in a new and powerful way, revealing new connections and structure, and allows you to “think bigger thoughts”.
- For **computer scientists**: category theory gives a precise handle on important notions such as compositionality, abstraction, representation-independence, genericity and more. Otherwise put, it provides the fundamental mathematical structures underpinning many key programming concepts.
- For **logicians**: category theory gives a syntax-independent view of the fundamental structures of logic, and opens up new kinds of models and interpretations.

- For **philosophers**: category theory opens up a fresh approach to structuralist foundations of mathematics and science; and an alternative to the traditional focus on set theory.
- For **physicists**: category theory offers new ways of formulating physical theories in a structural form. There have *inter alia* been some striking recent applications to quantum information and computation.

1.1.1 From Elements To Arrows

Category theory can be seen as a “generalised theory of functions”, where the focus is shifted from the pointwise, set-theoretic view of functions, to an abstract view of functions as *arrows*.

Let us briefly recall the arrow notation for functions between sets.¹ A function f with *domain* X and *codomain* Y is denoted by: $f : X \rightarrow Y$.

Diagrammatic notation: $X \xrightarrow{f} Y$.

The fundamental operation on functions is *composition*: if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then we can define $g \circ f : X \rightarrow Z$ by $g \circ f(x) := g(f(x))$.² Note that, in order for the composition to be defined, the codomain of f must be the same as the domain of g .

Diagrammatic notation: $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Moreover, for each set X there is an *identity function* on X , which is denoted by:

$$\text{id}_X : X \longrightarrow X \quad \text{id}_X(x) := x.$$

These operations are governed by the *associativity law* and the *unit laws*. For $f : X \rightarrow Y$, $g : Y \rightarrow Z$, $h : Z \rightarrow W$:

$$(h \circ g) \circ f = h \circ (g \circ f), \quad f \circ \text{id}_X = f = \text{id}_Y \circ f.$$

Notice that these equations are formulated purely in terms of the algebraic operations on functions, without any reference to the elements of the sets X , Y , Z , W . We will refer to any concept pertaining to functions which can be defined purely in terms of composition and identities as *arrow-theoretic*. We will now take a first step towards learning to “think with arrows” by seeing how we can replace some familiar definitions couched in terms of elements by arrow-theoretic equivalents; this will lead us towards the notion of category.

We say that a function $f : X \longrightarrow Y$ is:

¹ A review of basic ideas about sets, functions and relations, and some of the notation we will be using, is provided in Appendix A.

² We shall use the notation “:=” for “is defined to be” throughout these notes.

injective if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective if $\forall y \in Y. \exists x \in X. f(x) = y$,

monic if $\forall g, h. f \circ g = f \circ h \implies g = h$,
epic if $\forall g, h. g \circ f = h \circ f \implies g = h$.

Note that injectivity and surjectivity are formulated in terms of elements, while epic and monic are arrow-theoretic.

Proposition 1 *Let $f : X \rightarrow Y$. Then,*

1. *f is injective iff f is monic.*
2. *f is surjective iff f is epic.*

Proof: We show 1. Suppose $f : X \rightarrow Y$ is injective, and that $f \circ g = f \circ h$, where $g, h : Z \rightarrow X$. Then, for all $z \in Z$:

$$f(g(z)) = f \circ g(z) = f \circ h(z) = f(h(z)).$$

Since f is injective, this implies $g(z) = h(z)$. Hence we have shown that

$$\forall z \in Z. g(z) = h(z),$$

and so we can conclude that $g = h$. So f injective implies f monic.

For the converse, fix a one-element set $\mathbf{1} = \{\bullet\}$. Note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \rightarrow X$, where $\bar{x}(\bullet) := x$. Moreover, if $f(x) = y$ then $\bar{y} = f \circ \bar{x}$. Writing injectivity in these terms, it amounts to the following.

$$\forall x, x' \in X. f \circ \bar{x} = f \circ \bar{x}' \implies \bar{x} = \bar{x}'$$

Thus we see that being injective is a *special case* of being monic. ■

Exercise 2 Show that $f : X \rightarrow Y$ is surjective iff it is epic.

1.1.2 Categories Defined

Definition 3 A *category* \mathcal{C} consists of:

- A collection $\text{Ob}(\mathcal{C})$ of **objects**. Objects are denoted by A, B, C , etc.
- A collection $\text{Ar}(\mathcal{C})$ of **arrows** (or **morphisms**). Arrows are denoted by f, g, h , etc.
- Mappings $\text{dom}, \text{cod} : \text{Ar}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$, which assign to each arrow f its **domain** $\text{dom}(f)$ and its **codomain** $\text{cod}(f)$. An arrow f with domain A and codomain B is written $f : A \rightarrow B$. For each pair of objects A, B , we define the set

$$\mathcal{C}(A, B) := \{f \in \text{Ar}(\mathcal{C}) \mid f : A \rightarrow B\}.$$

We refer to $\mathcal{C}(A, B)$ as a **hom-set**. Note that distinct hom-sets are *disjoint*.

- For any triple of objects A, B, C , a **composition** map

$$c_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C).$$

$c_{A,B,C}(f, g)$ is written $g \circ f$ (or sometimes $f; g$). Diagrammatically:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

- For each object A , an **identity** arrow $\text{id}_A : A \rightarrow A$.

The above must satisfy the following axioms.

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad f \circ \text{id}_A = f = \text{id}_B \circ f.$$

whenever the domains and codomains of the arrows match appropriately so that the compositions are well-defined. ▲

1.1.3 Diagrams in Categories

Diagrammatic reasoning is an important tool in category theory. The basic cases are commuting triangles and squares. To say that the following triangle commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

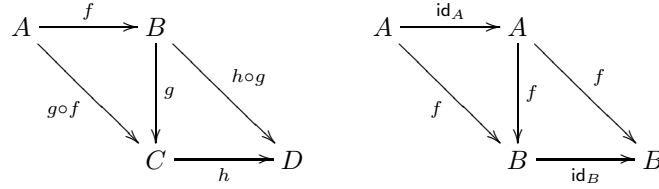
is exactly equivalent to asserting the equation $g \circ f = h$. Similarly, to say that the following square commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow h & & \downarrow g \\ C & \xrightarrow{k} & D \end{array}$$

means exactly that $g \circ f = k \circ h$. For example, the equations

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad f \circ \text{id}_A = f = \text{id}_B \circ f,$$

can be expressed by saying that the following diagrams commute.



As these examples illustrate, most of the diagrams we shall use will be “pasted together” from triangles and squares: the commutation of the diagram as a whole will then reduce to the commutation of the constituent triangles and squares.

We turn to the general case. The formal definition is slightly cumbersome; we give it anyway for reference.

Definition 4 We define a **graph** to be a collection of **vertices** and **directed edges**, where each edge $e : v \rightarrow w$ has a specified source vertex v and target vertex w . Thus graphs are like categories without composition and identities.³ A **diagram in a category \mathcal{C}** is a graph whose vertices are labelled with objects of \mathcal{C} and whose edges are labelled with arrows of \mathcal{C} , such that, if $e : v \rightarrow w$ is labelled with $f : A \rightarrow B$, then we must have v labelled by A and w labelled by B . We say that such a diagram *commutes* if any two paths in it with common source and target, and at least one of which has length greater than 1, are equal. That is, given paths

$$A \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots C_{n-1} \xrightarrow{f_n} B \quad \text{and} \quad A \xrightarrow{g_1} D_1 \xrightarrow{g_2} \cdots D_{m-1} \xrightarrow{g_m} B,$$

if $\max(n, m) > 1$ then

$$f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1.$$

▲

To illustrate this definition, to say that the following diagram commutes

$$E \xrightarrow{e} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

amounts to the assertion that $f \circ e = g \circ e$; it does *not* imply that $f = g$.

³ This would be a “multigraph” in normal parlance, since multiple edges between a given pair of vertices are allowed.

1.1.4 Examples

Before we proceed to our first examples of categories, we shall present some background material on partial orders, monoids and topologies, which will provide running examples throughout these notes.

Partial orders

A *partial order* is a structure (P, \leq) where P is a set and \leq is a binary relation on P satisfying:

- $x \leq x$ (Reflexivity)
- $x \leq y \wedge y \leq x \Rightarrow x = y$ (Antisymmetry)
- $x \leq y \wedge y \leq z \Rightarrow x \leq z$ (Transitivity)

For example, (\mathbb{R}, \leq) and $(\mathcal{P}(X), \subseteq)$ are partial orders, and so are strings with the sub-string relation.

If P, Q are partial orders, a map $h : P \rightarrow Q$ is a *partial order homomorphism* (or *monotone function*) if:

$$\forall x, y \in P. x \leq y \implies h(x) \leq h(y).$$

Note that homomorphisms are closed under composition, and that identity maps are homomorphisms.

Monoids

A *monoid* is a structure $(M, \cdot, 1)$ where M is a set,

$$\cdot : M \times M \longrightarrow M$$

is a binary operation, and $1 \in M$, satisfying the following axioms.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad 1 \cdot x = x = x \cdot 1.$$

For example, $(\mathbb{N}, +, 0)$ is a monoid, and so are strings with string-concatenation. Moreover, groups are special kinds of monoids.

If M, N are monoids, a map $h : M \rightarrow N$ is a *monoid homomorphism* if

$$\forall m_1, m_2 \in M. h(m_1 \cdot m_2) = h(m_1) \cdot h(m_2), \quad h(1) = 1.$$

Exercise 5 Suppose that G and H are groups (and hence monoids), and that $h : G \rightarrow H$ is a monoid homomorphism. Prove that h is a group homomorphism.

Topological spaces

A topological space is a pair (X, T_X) where X is a set, and T_X is a family of subsets of X such that

- $\emptyset, X \in T_X$,
- if $U, V \in T_X$ then $U \cap V \in T_X$,
- if $\{U_i\}_{i \in I}$ is any family in T_X , then $\bigcup_{i \in I} U_i \in T_X$.

A *continuous map* $f : (X, T_X) \rightarrow (Y, T_Y)$ is a function $f : X \rightarrow Y$ such that, for all $U \in T_Y$, $f^{-1}(U) \in T_X$.

Let us now see some first examples of categories.

- Any kind of mathematical structure, together with structure preserving functions, forms a category. E.g.
 - **Set** (sets and functions)
 - **Mon** (monoids and monoid homomorphisms)
 - **Grp** (groups and group homomorphisms)
 - **Vect_k** (vector spaces over a field k , and linear maps)
 - **Pos** (partially ordered sets and monotone functions)
 - **Top** (topological spaces and continuous functions)
- **Rel**: objects are sets, arrows $R : X \rightarrow Y$ are *relations* $R \subseteq X \times Y$.
Relational composition:

$$R; S(x, z) \iff \exists y. R(x, y) \wedge S(y, z)$$

- Let k be a field (for example, the real or complex numbers). Consider the following category **Mat_k**. The objects are natural numbers. A morphism $M : \mathbf{n} \rightarrow \mathbf{m}$ is an $\mathbf{n} \times \mathbf{m}$ matrix with entries in k . Composition is matrix multiplication, and the identity on \mathbf{n} is the $\mathbf{n} \times \mathbf{n}$ diagonal matrix.
- ◇ Monoids are one-object categories. Arrows correspond to the elements of the monoid, with the monoid operation being arrow-composition and the monoid unit being the identity arrow.
- ◇ A category in which for each pair of objects A, B there is at most one morphism from A to B is the same thing as a *preorder*, i.e. a reflexive and transitive relation.

Note that our first class of examples illustrate the idea of categories as *mathematical contexts*; settings in which various mathematical theories can be developed. Thus for example, **Top** is the context for general topology, **Grp** is the context for group theory, etc.

On the other hand, the last two examples illustrate that many important mathematical structures themselves appear as categories of particular kinds. The fact that two such different kinds of structures as monoids and posets should appear as extremal versions of categories is also rather striking.

This ability to capture mathematics both “in the large” and “in the small” is a first indication of the flexibility and power of categories.

Exercise 6 Check that **Mon**, **Vect_k**, **Pos** and **Top** are indeed categories.

Exercise 7 Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon** have?)

Exercise 8 Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

1.1.5 First Notions

Many important mathematical notions can be expressed at the general level of categories.

Definition 9 Let \mathcal{C} be a category. A morphism $f : X \rightarrow Y$ in \mathcal{C} is:

- *monic* (or a *monomorphism*) if $f \circ g = f \circ h \implies g = h$,
- *epic* (or an *epimorphism*) if $g \circ f = h \circ f \implies g = h$.

An *isomorphism* in \mathcal{C} is an arrow $i : A \rightarrow B$ such that there exists an arrow $j : B \rightarrow A$ — the *inverse* of i — satisfying

$$j \circ i = \text{id}_A, \quad i \circ j = \text{id}_B.$$

▲

We denote isomorphisms by $i : A \xrightarrow{\cong} B$, and write i^{-1} for the inverse of i . We say that A and B *are isomorphic*, $A \cong B$, if there exists some $i : A \xrightarrow{\cong} B$.

Exercise 10 Show that the inverse, if it exists, is unique.

Exercise 11 Show that \cong is an equivalence relation on the objects of a category.

As we saw previously, in **Set** monics are injections and epics are surjections. On the other hand, isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, in **Pos** to order isomorphisms, etc.

Exercise 12 Verify these claims.

Thus we have at one stroke captured the key notion of isomorphism in a form which applies to *all* mathematical contexts. This is a first taste of the level of generality which category theory naturally affords.

We have already identified monoids as one-object categories. We can now identify groups as *exactly those one-object categories in which every arrow is an isomorphism*. This also leads to a natural generalisation, of considerable importance in current mathematics: a **groupoid** is a category in which every morphism is an isomorphism.

Opposite Categories and Duality

The directionality of arrows within a category \mathcal{C} can be reversed without breaking the conditions of being a category; this yields the notion of **opposite category**.

Definition 13 Given a category \mathcal{C} , *the opposite category* \mathcal{C}^{op} is given by taking the same objects as \mathcal{C} , and

$$\mathcal{C}^{\text{op}}(A, B) := \mathcal{C}(B, A).$$

Composition and identities are inherited from \mathcal{C} . ▲

Note that if we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in \mathcal{C}^{op} , this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in \mathcal{C} , so composition $g \circ f$ in \mathcal{C}^{op} is defined as $f \circ g$ in \mathcal{C} !

Consideration of opposite categories leads to a **principle of duality**: a statement S is true about \mathcal{C} if and only if its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about \mathcal{C}^{op} . For example,

A morphism f is monic in \mathcal{C}^{op} if and only if it is epic in \mathcal{C} .

Indeed, f is monic in \mathcal{C}^{op} iff for all $g, h : C \rightarrow B$ in \mathcal{C}^{op} ,

$$f \circ g = f \circ h \implies g = h,$$

iff for all $g, h : B \rightarrow C$ in \mathcal{C} ,

$$g \circ f = h \circ f \implies g = h,$$

iff f is epic in \mathcal{C} . We say that monic and epic are *dual notions*.

Exercise 14 If P is a preorder, for example (\mathbb{R}, \leq) , describe P^{op} explicitly.

Subcategories

Another way to obtain new categories from old ones is by restricting their objects or arrows.

Definition 15 Let \mathcal{C} be a category. Suppose that we are given collections

$$\text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C}), \quad \forall A, B \in \text{Ob}(\mathcal{D}). \mathcal{D}(A, B) \subseteq \mathcal{C}(A, B).$$

We say that \mathcal{D} is a **subcategory** of \mathcal{C} if

$$A \in \text{Ob}(\mathcal{D}) \Rightarrow \text{id}_A \in \mathcal{D}(A, A), \quad f \in \mathcal{D}(A, B), g \in \mathcal{D}(B, C) \Rightarrow g \circ f \in \mathcal{D}(A, C),$$

and hence \mathcal{D} itself is a category. In particular, \mathcal{D} is:

- A **full** subcategory of \mathcal{C} if for any $A, B \in \text{Ob}(\mathcal{D})$, $\mathcal{D}(A, B) = \mathcal{C}(A, B)$.
- A **lluf** subcategory of \mathcal{C} if $\text{Ob}(\mathcal{D}) = \text{Ob}(\mathcal{C})$. ▲

For example, **Grp** is a full subcategory of **Mon** (by Exercise 5), and **Set** is a lluf subcategory of **Rel**.

Simple cats

We close this section with some very basic examples of categories.

- $\mathbb{1}$ is the category with one object and one arrow, that is,

$$\mathbb{1} := \bullet \begin{array}{c} \circ \\ \curvearrowright \end{array}$$

where the arrow is necessarily id_\bullet . Note that, although we say that $\mathbb{1}$ is *the* one-object/one-arrow category, there is by no means a *unique* such category. This is explained by the intuitively evident fact that any two such categories are isomorphic. (We will define what it means for categories to be isomorphic later.)

- In two-object categories, there is the one with two arrows, $\mathbb{2} := \bullet \bullet$, and also:

$$\mathbb{2}_\rightarrow := \bullet \longrightarrow \bullet, \quad \mathbb{2}_\rightrightarrows := \bullet \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \bullet, \quad \mathbb{2}_\rightleftarrows := \bullet \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \bullet \quad \dots$$

Note that we have omitted identity arrows for economy. Categories with only identity arrows, like $\mathbb{1}$ and $\mathbb{2}$, are called *discrete categories*.

Exercise 16 How many categories \mathcal{C} with $\text{Ob}(\mathcal{C}) = \{\bullet\}$ are there? (Hint: what do such categories correspond to?)

1.1.6 Exercises

1. Consider the following properties of an arrow f in a category \mathcal{C} .
 - f is *split monic* if for some g , $g \circ f$ is an identity arrow.
 - f is *split epic* if for some g , $f \circ g$ is an identity arrow.
 - a. Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.
 - b. Prove that, if f is split epic then it is epic.
 - c. Prove that, if f and $g \circ f$ are iso then g is iso.
 - d. Prove that, if f is monic and split epic then it is iso.
 - e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map

$$i : (\mathbb{N}, +, 0) \longrightarrow (\mathbb{Z}, +, 0)$$
 of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that, if $\{X_i\}_{i \in I}$ is a family of non-empty sets, we can form a set $X = \{x_i \mid i \in I\}$ where $x_i \in X_i$ for all $i \in I$.

 - f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
 - g. Is it always the case that an arrow which is epic is split epic? Either prove that it is, or give a counter-example.
2. Give a description of partial orders as categories of a special kind.

1.2 Some Basic Constructions

We shall now look at a number of basic constructions which appear throughout mathematics, and which acquire their proper general form in the language of categories.

1.2.1 Initial and Terminal Objects

A first such example is that of initial and terminal objects. While apparently trivial, they are actually both important and useful, as we shall see in the sequel.

Definition 17 An object I in a category \mathcal{C} is *initial* if, for every object A , there exists a unique arrow from I to A , which we write $\iota_A : I \rightarrow A$.

A **terminal** object in \mathcal{C} is an object T such that, for every object A , *there exists a unique arrow* from A to T , which we write $\tau_A : A \rightarrow T$. \blacktriangle

Note that initial and terminal objects are dual notions: T is terminal in \mathcal{C} iff it is initial in \mathcal{C}^{op} . We sometimes write $\mathbf{1}$ for the terminal object and $\mathbf{0}$ for the initial one. Note also the assertions of *unique existence* in the definitions. This is one of the *leitmotifs* of category theory; we shall encounter it again in a conceptually deeper form in section 1.5.

Let us examine initial and terminal objects in our standard example categories.

- In **Set**, the empty set is an initial object while any one-element set $\{\bullet\}$ is terminal.
- In **Pos**, the poset (\emptyset, \emptyset) is an initial object while $(\{\bullet\}, \{(\bullet, \bullet)\})$ is terminal.
- In **Top**, the space $(\emptyset, \{\emptyset\})$ is an initial object while $(\{\bullet\}, \{\emptyset, \{\bullet\}\})$ is terminal.
- In **Vect_k**, the one-element space $\{0\}$ is both initial and terminal.
- In a poset, seen as a category, an initial object is a least element, while a terminal object is a greatest element.

Exercise 18 Verify these claims. In each case, identify the canonical arrows.

Exercise 19 Identify the initial and terminal objects in **Rel**.

Exercise 20 Suppose that a monoid, viewed as a category, has either an initial or a terminal object. What must the monoid be?

We shall now establish a fundamental fact: initial and terminal objects are *unique up to (unique) isomorphism*. As we shall see, this is characteristic of all such “universal” definitions. For example, the apparent arbitrariness in the fact that any singleton set is a terminal object in **Set** is answered by the fact that what counts is the property of being terminal; and this suffices to ensure that any two concrete objects having this property must be isomorphic to each other.

The proof of the proposition, while elementary, is a first example of distinctively categorical reasoning.

Proposition 21 *If I and I' are initial objects in the category \mathcal{C} then there exists a unique isomorphism $I \xrightarrow{\cong} I'$.*

Proof: Since I is initial and I' is an object of \mathcal{C} , there is a unique arrow $\iota_{I'} : I \rightarrow I'$. We claim that $\iota_{I'}$ is an isomorphism.

Since I' is initial and I is an object in \mathcal{C} , there is an arrow $\iota'_I : I' \rightarrow I$. Thus we obtain $\iota_{I'}; \iota'_I : I \rightarrow I$, while we also have the identity morphism $\text{id}_I : I \rightarrow I$. But I is initial and therefore there exists a *unique* arrow from I to I , which means that $\iota_{I'}; \iota'_I = \text{id}_I$. Similarly, $\iota'_I; \iota_{I'} = \text{id}_{I'}$, so $\iota_{I'}$ is indeed an isomorphism. \blacksquare

Hence, initial objects are “unique up to (unique) isomorphism”, and we can (and do) speak of *the* initial object (if any such exists). Similarly for terminal objects.

Exercise 22 Let \mathcal{C} be a category with an initial object $\mathbf{0}$. For any object A , show the following.

- If $A \cong \mathbf{0}$ then A is an initial object.
- If there exists a monomorphism $f : A \rightarrow \mathbf{0}$ then f is an iso, and hence A is initial.

1.2.2 Products and Coproducts

1.2.2.1 Products

We now consider one of the most common constructions in mathematics: the formation of “direct products”. Once again, rather than giving a case-by-case construction of direct products in each mathematical context we encounter, we can express once and for all a general notion of product, meaningful in any category — and such that, if a product exists, it is characterised uniquely up to unique isomorphism, just as for initial and terminal objects. Given a particular mathematical context, *i.e.* a category, we can then verify whether or not the product exists in that category. The concrete construction appropriate to the context will enter only into the proof of *existence*; all of the useful *properties* of the product follow from the general definition. Moreover, the categorical notion of product has a *normative* force; we can test whether a concrete construction works as intended by verifying that it satisfies the general definition.

In set theory, the cartesian product is defined in terms of the ordered pair:

$$X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}.$$

It turns out that ordered pairs can be *defined* in set theory, e.g. as

$$(x, y) := \{\{x, y\}, y\}.$$

Note that in no sense is such a definition canonical. The essential *properties* of ordered pairs are:

1. We can retrieve the first and second components x, y of the ordered pair (x, y) , allowing *projection functions* to be defined:

$$\pi_1 : (x, y) \mapsto x, \quad \pi_2 : (x, y) \mapsto y.$$

2. The information about first and second components completely determines the ordered pair:

$$(x_1, x_2) = (y_1, y_2) \iff x_1 = y_1 \wedge x_2 = y_2.$$

The categorical definition expresses these properties in arrow-theoretic terms, meaningful in any category.

Definition 23 Let A, B be objects in a category \mathcal{C} . An A, B -pairing is a triple (P, p_1, p_2) where P is an object, $p_1 : P \rightarrow A$ and $p_2 : P \rightarrow B$. A morphism of A, B -pairings

$$f : (P, p_1, p_2) \longrightarrow (Q, q_1, q_2)$$

is a morphism $f : P \rightarrow Q$ in \mathcal{C} such that $q_1 \circ f = p_1$ and $q_2 \circ f = p_2$, i.e. the following diagram commutes.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow p_1 & \downarrow f & \searrow p_2 & \\ A & \xleftarrow{q_1} & Q & \xrightarrow{q_2} & B \end{array}$$

The A, B -pairings form a category $\mathbf{Pair}(A, B)$. We say that $(A \times B, \pi_1, \pi_2)$ is a **product** of A and B if it is *terminal* in $\mathbf{Pair}(A, B)$. \blacktriangle

Exercise 24 Verify that $\mathbf{Pair}(A, B)$ is a category.

Note that products are specified by triples $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$, where π_i 's are called *projections*. For economy (and if projections are obvious) we may say that $A \times B$ is the product of A and B . We say that \mathcal{C} **has (binary) products** if each pair of objects A, B has a product in \mathcal{C} . A direct consequence of the definition, by Proposition 21, is that if products exist, they are unique up to (unique) isomorphism.

Unpacking the uniqueness condition from $\mathbf{Pair}(A, B)$ back to \mathcal{C} we obtain a more concise definition of products which we use in practice.

Definition 25 (Equivalent definition of product) Let A, B be objects in a category \mathcal{C} . A product of A and B is an object $A \times B$ together with a pair of arrows $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ such that for every triple $A \xleftarrow{f} C \xrightarrow{g} B$ there exists a *unique* morphism

$$\langle f, g \rangle : C \longrightarrow A \times B$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\
& \searrow f & \uparrow \langle f, g \rangle & \nearrow g & \\
& & C & &
\end{array}
\quad \left(\begin{array}{l} \pi_1 \circ \langle f, g \rangle = f \\ \pi_2 \circ \langle f, g \rangle = g \end{array} \right)$$

▲

We call $\langle f, g \rangle$ the *pairing* of f and g .

Note that the above diagram features a *dashed arrow*. Our intention with such diagrams is always to express the following idea: if the undashed part of the diagram commutes, then *there exists a unique arrow* (the dashed one) such that the whole diagram commutes. In any case, we shall always spell out the intended statement explicitly.

We look at how this definition works in our standard example categories.

- In **Set**, products are the usual cartesian products.
- In **Pos**, products are cartesian products with the pointwise order.
- In **Top**, products are cartesian products with the product topology.
- In **Vect_k**, products are direct sums.
- In a poset, seen as a category, products are *greatest lower bounds*.

Exercise 26 Verify these claims.

The following proposition shows that the uniqueness of the pairing arrow can be specified purely equationally, by the equation:

$$\forall h : C \rightarrow A \times B. h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$$

Proposition 27 *For any triple $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ the following statements are equivalent.*

- (I) *For any triple $A \xleftarrow{f} C \xrightarrow{g} B$ there exists a unique morphism $\langle f, g \rangle : C \rightarrow A \times B$ such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.*
- (II) *For any triple $A \xleftarrow{f} C \xrightarrow{g} B$ there exists a morphism $\langle f, g \rangle : C \rightarrow A \times B$ such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$, and moreover, for any $h : C \rightarrow A \times B$, $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$.*

Proof: For (I) \Rightarrow (II), take any $h : C \rightarrow A \times B$; we need to show $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$. We have

$$A \xleftarrow{\pi_1 \circ h} C \xrightarrow{\pi_2 \circ h} B$$

and hence, by (I), there exists unique $k : C \rightarrow A \times B$ such that

$$\pi_1 \circ k = \pi_1 \circ h \quad \wedge \quad \pi_2 \circ k = \pi_2 \circ h \quad (*)$$

Note now that $(*)$ holds both for $k := h$ and $k := \langle \pi_1 \circ h, \pi_2 \circ h \rangle$, the latter because of (I). Hence, $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$.

For (II) \Rightarrow (I), take any triple $A \xleftarrow{f} C \xrightarrow{g} B$. By (II), we have that there exists an arrow $\langle f, g \rangle : C \rightarrow A \times B$ such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$. We need to show it is the unique such. Let $k : C \rightarrow A \times B$ s.t.

$$\pi_1 \circ k = f \quad \wedge \quad \pi_2 \circ k = g$$

Then, by (II),

$$k = \langle \pi_1 \circ k, \pi_2 \circ k \rangle = \langle f, g \rangle$$

as required. \blacksquare

In the following proposition we give some useful properties of products. First, let us introduce some notation for arrows: given $f_1 : A_1 \rightarrow B_1$, $f_2 : A_2 \rightarrow B_2$, define

$$f_1 \times f_2 := \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle : A_1 \times A_2 \longrightarrow B_1 \times B_2.$$

Proposition 28 *For any $f : A \rightarrow B$, $g : A \rightarrow C$, $h : A' \rightarrow A$, and any $p : B \rightarrow B'$, $q : C \rightarrow C'$,*

- $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$,
- $(p \times q) \circ \langle f, g \rangle = \langle p \circ f, q \circ g \rangle$.

Proof: For the first claim we have:

$$\langle f, g \rangle \circ h = \langle \pi_1 \circ (\langle f, g \rangle \circ h), \pi_2 \circ (\langle f, g \rangle \circ h) \rangle = \langle f \circ h, g \circ h \rangle.$$

And for the second:

$$\begin{aligned} (p \times q) \circ \langle f, g \rangle &= \langle p \circ \pi_1, q \circ \pi_2 \rangle \circ \langle f, g \rangle \\ &= \langle p \circ \pi_1 \circ \langle f, g \rangle, q \circ \pi_2 \circ \langle f, g \rangle \rangle \\ &= \langle p \circ f, q \circ g \rangle. \end{aligned} \quad \blacksquare$$

General Products

The notion of products can be generalised to arbitrary arities as follows. A product for a family of objects $\{A_i\}_{i \in I}$ in a category \mathcal{C} is an object P and morphisms

$$p_i : P \longrightarrow A_i \quad (i \in I)$$

such that, for all objects B and arrows

$$f_i : B \longrightarrow A_i \quad (i \in I)$$

there is a *unique* arrow

$$g : B \longrightarrow P$$

such that, for all $i \in I$, the following diagram commutes.

$$\begin{array}{ccc}
 B & \xrightarrow{g} & P \\
 & \searrow f_i \quad \swarrow p_i & \\
 & A_i &
 \end{array}$$

As before, if such a product exists, it is unique up to (unique) isomorphism. We write $P = \prod_{i \in I} A_i$ for the product object, and $g = \langle f_i \mid i \in I \rangle$ for the unique morphism in the definition.

Exercise 29 What is the product of the empty family?

Exercise 30 Show that if a category has binary and nullary products then it has all finite products.

1.2.2.2 Coproducts

We now investigate the dual notion to products: namely coproducts. Formally, coproducts in \mathcal{C} are just products in \mathcal{C}^{op} , interpreted back in \mathcal{C} . We spell out the definition.

Definition 31 Let A, B be objects in a category \mathcal{C} . A *coproduct* of A and B is an object $A + B$ together with a pair of arrows $A \xrightarrow{\text{in}_1} A + B \xleftarrow{\text{in}_2} B$ such that for every triple $A \xrightarrow{f} C \xleftarrow{g} B$ there exists a *unique* morphism

$$[f, g] : A + B \longrightarrow C$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
 A & \xrightarrow{\text{in}_1} & A + B & \xleftarrow{\text{in}_2} & B \\
 & \searrow f & \downarrow [f, g] & \swarrow g & \\
 & & C & &
 \end{array}
 \quad \left(\begin{array}{l} [f, g] \circ \text{in}_1 = f \\ [f, g] \circ \text{in}_2 = g \end{array} \right)$$

▲

We call the in_i 's *injections* and $[f, g]$ the *copairing* of f and g . As with pairings, uniqueness of copairings can be specified by an equation:

$$\forall h : A + B \rightarrow C. h = [h \circ \text{in}_1, h \circ \text{in}_2]$$

Coproducts in **Set**

This is given by *disjoint union* of sets, which can be defined concretely e.g. by

$$X + Y := \{1\} \times X \cup \{2\} \times Y.$$

We can define *injections*

$$X \xrightarrow{\text{in}_1} X + Y \xleftarrow{\text{in}_2} Y$$

$$\text{in}_1(x) := (1, x), \quad \text{in}_2(y) := (2, y).$$

Also, given functions $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, we can define

$$[f, g] : X + Y \rightarrow Z$$

$$[f, g](1, x) := f(x), \quad [f, g](2, y) := g(y).$$

Exercise 32 Check that this construction does yield coproducts in **Set**.

Note that this example suggests that coproducts allow for *definition by cases*.

Let us examine coproducts for some of our other standard examples.

- In **Pos**, disjoint unions (with the inherited orders) are coproducts.
- In **Top**, topological disjoint unions are coproducts.
- In **Vect_k**, direct sums are coproducts.
- In a poset, *least upper bounds* are coproducts.

Exercise 33 Verify these claims.

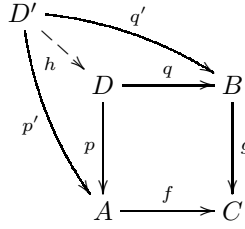
Exercise 34 Dually to products, express coproducts as initial objects of a category **Copair**(A, B) of A, B -copairings.

1.2.3 Pullbacks and Equalisers

We shall consider two further constructions of interest: *pullbacks* and *equalisers*.

1.2.3.1 Pullbacks

Definition 35 Consider a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$. The **pullback** of f along g is a pair $A \xleftarrow{p} D \xrightarrow{q} B$ such that $f \circ p = g \circ q$ and, for any pair $A \xleftarrow{p'} D' \xrightarrow{q'} B$ such that $f \circ p' = g \circ q'$, there exists a unique $h : D' \rightarrow D$ such that $p' = p \circ h$ and $q' = q \circ h$. Diagrammatically,



▲

Example 36

- In **Set** the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is defined as a *subset of the cartesian product*:

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

For example, consider a category \mathcal{C} with

$$\text{Ar}(\mathcal{C}) \xrightarrow{\text{dom}} \text{Ob}(\mathcal{C}) \xleftarrow{\text{cod}} \text{Ar}(\mathcal{C}).$$

Then the pullback of dom along cod is the set of *composable morphisms*, *i.e.* pairs of morphisms (f, g) in \mathcal{C} such that $f \circ g$ is well-defined.

- In **Set** again, subsets (*i.e.* inclusion maps) pull back to subsets:

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Exercise 37 Let \mathcal{C} be a category with a terminal object $\mathbf{1}$. Show that, for any $A, B \in \text{Ob}(\mathcal{C})$, the pullback of $A \xrightarrow{\tau_A} \mathbf{1} \xleftarrow{\tau_B} B$ is the product of A and B , if it exists.

Just as for products, pullbacks can equivalently be described as terminal objects in suitable categories. Given a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$, we define an (f, g) -*cone* to be a triple (D, p, q) such that the following diagram commutes.

$$\begin{array}{ccc} D & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

A morphism of (f, g) -cones $h : (D_1, p_1, q_1) \rightarrow (D_2, p_2, q_2)$ is a morphism $h : D_1 \rightarrow D_2$ such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & D_1 & & \\
 & \swarrow p_1 & \downarrow h & \searrow q_1 & \\
 A & \xleftarrow{p_2} & D_2 & \xrightarrow{q_2} & B
 \end{array}$$

We can thus form a category $\mathbf{Cone}(f, g)$. A pull-back of f along g , if it exists, is exactly a terminal object of $\mathbf{Cone}(f, g)$. Once again, this shows the uniqueness of pullbacks up to unique isomorphism.

1.2.3.2 Equalisers

Definition 38 Consider a pair of parallel arrows $A \xrightleftharpoons[g]{f} B$. An *equaliser* of (f, g) is an arrow $e : E \rightarrow A$ such that $f \circ e = g \circ e$ and, for any arrow $h : D \rightarrow A$ such that $f \circ h = g \circ h$, there is a unique $\hat{h} : D \rightarrow E$ so that $h = e \circ \hat{h}$. Diagrammatically,

$$\begin{array}{ccccc}
 E & \xrightarrow{e} & A & \xrightleftharpoons[g]{f} & B \\
 \uparrow \hat{h} & \nearrow h & & & \\
 D & & & &
 \end{array}$$

▲

As for products, uniqueness of the arrow from D to E can be expressed equationally:

$$\forall k : D \rightarrow E. \widehat{e \circ k} = k.$$

Exercise 39 Why is $\widehat{e \circ k}$ well-defined for any $k : D \rightarrow E$? Prove that the above equation is equivalent to the uniqueness requirement.

Example 40 In \mathbf{Set} , the equaliser of f, g is given by the inclusion

$$\{x \in A \mid f(x) = g(x)\} \hookrightarrow A.$$

This allows *equationally defined subsets* to be defined as equalisers. For example, consider the pair of maps $\mathbb{R}^2 \xrightleftharpoons[g]{f} \mathbb{R}$, where

$$f : (x, y) \mapsto x^2 + y^2, \quad g : (x, y) \mapsto 1.$$

Then, the equaliser is the unit circle as a subset of \mathbb{R}^2 .

1.2.4 Limits and Colimits

The notions we have introduced so far are all special cases of a general notion of *limits* in categories, and the dual notion of *colimits*.

Limits	Colimits
Terminal Objects	Initial Objects
Products	Coproducts
Pullbacks	Pushouts
Equalisers	Coequalisers

Table 1.1: Examples of Limits and Colimits.

An important aspect of studying any kind of mathematical structure is to see what limits and colimits the category of such structures has. We shall return to these ideas shortly.

1.2.5 Exercises

1. Give an example of a category where some pair of objects lacks a product or coproduct.
2. (*Pullback lemma*) Consider the following commutative diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \downarrow u & & \downarrow v & & \downarrow w \\
 D & \xrightarrow{h} & E & \xrightarrow{i} & F
 \end{array}$$

Given that the right hand square $BCEF$ and the outer square $ACDF$ are pullbacks, prove that the left hand square $ABDE$ is a pullback.

3. Consider $A \xrightarrow{f} C \xleftarrow{g} B$ with pullback $A \xleftarrow{p} D \xrightarrow{q} B$. For each $A \xleftarrow{p'} D' \xrightarrow{q'} B'$ with $f \circ p' = g \circ q'$, let $\phi(p', q') : D' \rightarrow D$ be the arrow dictated by the pullback condition. Express uniqueness of $\phi(p', q')$ equationally.

1.3 Functors

Part of the “categorical philosophy” is:

Don't just look at the objects; take the morphisms into account too.

We can also apply this to categories!

1.3.1 Basics

A “morphism of categories” is a *functor*.

Definition 41 A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is given by:

- An object-map, assigning an object FA of \mathcal{D} to every object A of \mathcal{C} .
- An arrow-map, assigning an arrow $Ff : FA \rightarrow FB$ of \mathcal{D} to every arrow $f : A \rightarrow B$ of \mathcal{C} , in such a way that composition and identities are preserved:

$$F(g \circ f) = Fg \circ Ff, \quad F\mathrm{id}_A = \mathrm{id}_{FA}.$$

▲

Note that we use the same symbol to denote the object- and arrow-maps; in practice, this never causes confusion. Since functors preserve domains and codomains of arrows, for each pair of objects A, B of \mathcal{C} , there is a well-defined map

$$F_{A,B} : \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB).$$

The conditions expressing preservation of composition and identities are called *functoriality*.

Example 42 Let $(P, \leq), (Q, \leq)$ be preorders (seen as categories). A functor $F : (P, \leq) \rightarrow (Q, \leq)$ is specified by an object-map, say $F : P \rightarrow Q$, and an appropriate arrow-map. The arrow-map corresponds to the condition

$$\forall p_1, p_2 \in P. p_1 \leq p_2 \implies F(p_1) \leq F(p_2),$$

i.e. to monotonicity of F . Moreover, the functoriality conditions are trivial since in the codomain (Q, \leq) all hom-sets are singletons.

Hence, a functor between preorders is just a monotone map.

Example 43 Let $(M, \cdot, 1), (N, \cdot, 1)$ be monoids. A functor $F : (M, \cdot, 1) \rightarrow (N, \cdot, 1)$ is specified by a trivial object map (monoids are categories with a single object) and an arrow-map, say $F : M \rightarrow N$. The functoriality conditions correspond to

$$\forall m_1, m_2 \in M. F(m_1 \cdot m_2) = F(m_1) \cdot F(m_2), \quad F(1) = 1,$$

i.e. to F being a monoid homomorphism.

Hence, a functor between monoids is just a monoid homomorphism.

Other examples are the following.

- Inclusion of a sub-category, $\mathcal{C} \hookrightarrow \mathcal{D}$, is a functor (by taking the identity map for object- and arrow-map).
- The *covariant* powerset functor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$:

$$X \mapsto \mathcal{P}(X), \quad (f : X \rightarrow Y) \mapsto \mathcal{P}(f) := S \mapsto \{f(x) \mid x \in S\}.$$

- $U : \mathbf{Mon} \rightarrow \mathbf{Set}$ is the ‘forgetful’ or ‘underlying’ functor which sends a monoid to its set of elements, ‘forgetting’ the algebraic structure, and sends a homomorphism to the corresponding function between sets. There are similar forgetful functors for other categories of structured sets. Why are these trivial-looking functors useful? — We shall see!
- Group theory examples. The assignment of the commutator sub-group of a group extends to a functor from **Group** to **Group**; and the assignment of the quotient by this normal subgroup extends to a functor from **Group** to **AbGroup**. The assignment of the centraliser of a group does not!
- More sophisticated examples: e.g. *homology*. The basic idea of algebraic topology is that there are functorial assignments of algebraic objects (e.g. groups) to topological spaces, and variants of this idea (‘(co)homology theories’) are pervasive throughout modern pure mathematics.

Functors ‘of several variables’

We can generalise the notion of a functor to a mapping from several domain categories to a codomain category. For this we need the following definition.

Definition 44 For categories \mathcal{C}, \mathcal{D} define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows. An object in $\mathcal{C} \times \mathcal{D}$ is a pair of objects from \mathcal{C} and \mathcal{D} , and an arrow in $\mathcal{C} \times \mathcal{D}$ is a pair of arrows from \mathcal{C} and \mathcal{D} . Identities and arrow composition are defined componentwise:

$$\mathrm{id}_{(A,B)} := (\mathrm{id}_A, \mathrm{id}_B), \quad (f, g) \circ (f', g') := (f \circ f', g \circ g').$$

▲

A functor ‘of two variables’, with domains \mathcal{C} and \mathcal{D} , to \mathcal{E} is simply a functor:

$$F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}.$$

For example, there are evident projection functors

$$\mathcal{C} \longleftarrow \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}.$$

1.3.2 Further Examples

Set-valued functors

Many important constructions arise as functors $F : \mathcal{C} \rightarrow \mathbf{Set}$. For example:

- If G is a group, a functor $F : G \rightarrow \mathbf{Set}$ is an *action of G on a set*.
- If P is a poset representing time, a functor $F : P \rightarrow \mathbf{Set}$ is a notion of *set varying through time*. This is related to *Kripke semantics*, and to *forcing arguments* in set theory.
- Recall that 2_{\Rightarrow} is the category $\bullet \rightrightarrows \bullet$. Then, functors $F : 2_{\Rightarrow} \rightarrow \mathbf{Set}$ correspond to *directed graphs* understood as in Definition 4, *i.e.* as structures (V, E, s, t) , where V is a set of vertices, E is a set of edges, and $s, t : E \rightarrow V$ specify the source and target vertices for each edge.

Let us examine the first example in more detail. For a group $(G, \cdot, 1)$, a functor $F : G \rightarrow \mathbf{Set}$ is specified by a set X (to which the unique object of G is mapped), and by an arrow-map sending each element m of G to an endofunction on X , say $m \bullet - : X \rightarrow X$. Then, functoriality amounts to the conditions

$$\forall m_1, m_2 \in G. F(m_1 \cdot m_2) = F(m_1) \circ F(m_2), \quad F(1) = \text{id}_X,$$

that is, for all $m_1, m_2 \in G$ and all $x \in X$,

$$(m_1 \cdot m_2) \bullet x = m_1 \bullet m_2 \bullet x, \quad 1 \bullet x = x.$$

We therefore see that F defines an action of G on X .

Exercise 45 Verify that functors $F : 2_{\Rightarrow} \rightarrow \mathbf{Set}$ correspond to directed graphs.

Example: Lists

Data-type constructors are functors. As a basic example, we consider lists. There is a functor

$$\text{List} : \mathbf{Set} \longrightarrow \mathbf{Set}$$

which takes a set X to the set of all finite lists (sequences) of elements of X . **List** is functorial: its action on morphisms (*i.e.* functions, *i.e.* (functional) programs) is given by *maplist*:

$$\frac{f : X \longrightarrow Y}{\text{List}(f) : \text{List}(X) \longrightarrow \text{List}(Y)}$$

$$\text{List}(f)[x_1, \dots, x_n] := [f(x_1), \dots, f(x_n)]$$

We can upgrade **List** to a functor $\mathbf{MList} : \mathbf{Set} \rightarrow \mathbf{Mon}$ by mapping each set X to the monoid $(\mathbf{List}(X), *, \epsilon)$ and $f : X \rightarrow Y$ to $\mathbf{List}(f)$, as above. The monoid operation $*$: $\mathbf{List}(X) \times \mathbf{List}(X) \rightarrow \mathbf{List}(X)$ is list concatenation, and ϵ is the empty list. We call $\mathbf{MList}(X)$ the *free monoid* over X . This terminology will be justified in Chapter 5.

Products as functors

If a category \mathcal{C} has binary products, then there is automatically a functor

$$_ - \times _ - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

which takes each pair (A, B) to the product $A \times B$, and each (f, g) to

$$f \times g := \langle f \circ \pi_1, g \circ \pi_2 \rangle.$$

Functoriality is shown as follows, using proposition 28 and uniqueness of pairings in its equational form.

$$\begin{aligned} (f \times g) \circ (f' \times g') &= (f \times g) \circ \langle f' \circ \pi_1, g' \circ \pi_2 \rangle = \langle f \circ f' \circ \pi_1, g \circ g' \circ \pi_2 \rangle \\ &= (f \circ f') \times (g \circ g'), \\ \mathbf{id}_A \times \mathbf{id}_B &= \langle \mathbf{id}_A \circ \pi_1, \mathbf{id}_B \circ \pi_2 \rangle = \langle \pi_1 \circ \mathbf{id}_{A \times B}, \pi_2 \circ \mathbf{id}_{A \times B} \rangle = \mathbf{id}_{A \times B}. \end{aligned}$$

The category of categories

There is a category **Cat** whose objects are categories, and whose arrows are functors. Identities in **Cat** are given by identity functors:

$$\mathbf{id}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C} := A \mapsto A, f \mapsto f.$$

Composition of functors is defined in the evident fashion. Note that if $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ then, for $f : A \rightarrow B$ in \mathcal{C} ,

$$G \circ F(f) := G(F(f)) : G(F(A)) \longrightarrow G(F(B))$$

so the types work out. A category of categories sounds (and is) circular, but in practice is harmless: one usually makes some size restriction on the categories, and then **Cat** will be too ‘big’ to be an object of itself. See Appendix A.

Note that product categories are products in **Cat**! For any pair of categories \mathcal{C}, \mathcal{D} , set

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$$

where $\mathcal{C} \times \mathcal{D}$ the product category (defined previously) and π_i ’s the obvious projection functors. For any pair of functors $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$, set

$$\langle F, G \rangle : \mathcal{E} \longrightarrow \mathcal{C} \times \mathcal{D} := A \mapsto (FA, GA), \quad f \mapsto (Ff, Gf).$$

It is easy to see that $\langle F, G \rangle$ is indeed a functor. Moreover, satisfaction of the product diagram and uniqueness are shown exactly as in **Set**.

1.3.3 Contravariance

By definition, the arrow-map of a functor F is *covariant*: it preserves the direction of arrows, so if $f : A \rightarrow B$ then $Ff : FA \rightarrow FB$. A *contravariant* functor G does exactly the opposite: it reverses arrow-direction, so if $f : A \rightarrow B$ then $Gf : GB \rightarrow GA$. A concise way to express contravariance is as follows.

Definition 46 Let \mathcal{C}, \mathcal{D} be categories. A **contravariant** functor G from \mathcal{C} to \mathcal{D} is a functor $G : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. (Equivalently, a functor $G : \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$.) ▲

Explicitly, a contravariant functor G is given by an assignment of:

- an object GA in \mathcal{D} to every object A in \mathcal{C} ,
- an arrow $Gf : GB \rightarrow GA$ in \mathcal{D} to every arrow $f : A \rightarrow B$ in \mathcal{C} , such that (notice the change of order in composition):

$$G(g \circ f) = Gf \circ Gg, \quad G\text{id}_A = \text{id}_{GA}.$$

Note that functors of several variables can be covariant in some variables and contravariant in others, e.g.

$$F : \mathcal{C}^{\text{op}} \times \mathcal{D} \longrightarrow \mathcal{E}.$$

Examples of Contravariant Functors

- The contravariant powerset functor, $\mathcal{P}^{\text{op}} : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$, is given by:

$$\begin{aligned} \mathcal{P}^{\text{op}}(X) &:= \mathcal{P}(X). \\ \mathcal{P}^{\text{op}}(f : X \rightarrow Y) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) := T \mapsto \{x \in X \mid f(x) \in T\}. \end{aligned}$$

- The dual space functor on vector spaces:

$$(-)^* : \mathbf{Vect}_k^{\text{op}} \longrightarrow \mathbf{Vect}_k := V \mapsto V^*.$$

Note that these are both examples of the following idea: send an object A into functions from A into some fixed object. For example, the powerset can be written as $\mathcal{P}(X) = 2^X$, where we think of a subset in terms of its characteristic function.

Hom-functors

We now consider some fundamental examples of **Set**-valued functors. Given a category \mathcal{C} and an object A of \mathcal{C} , two functors to **Set** can be defined:

- The covariant Hom-functor at A ,

$$\mathcal{C}(A, _): \mathcal{C} \longrightarrow \mathbf{Set},$$

which is given by (recall that each $\mathcal{C}(A, B)$ is a set):

$$\mathcal{C}(A, _)(B) := \mathcal{C}(A, B), \quad \mathcal{C}(A, _)(f: B \rightarrow C) := g \mapsto f \circ g.$$

We usually write $\mathcal{C}(A, _)(f)$ as $\mathcal{C}(A, f)$. Functoriality reduces directly to the basic category axioms: associativity of composition and the unit laws for the identity.

- There is also a contravariant Hom-functor,

$$\mathcal{C}(_, A): \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set},$$

given by:

$$\mathcal{C}(_, A)(B) := \mathcal{C}(B, A), \quad \mathcal{C}(_, A)(h: C \rightarrow B) := g \mapsto g \circ h.$$

Generalising both of the above, we obtain a **bivariant** Hom-functor,

$$\mathcal{C}(_, _): \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}.$$

Exercise 47 Spell out the definition of $\mathcal{C}(_, _): \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$. Verify carefully that it is a functor.

1.3.4 Properties of Functors

Definition 48 A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be:

- **faithful** if each map $F_{A,B}: \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$ is injective;
- **full** if each map $F_{A,B}: \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$ is surjective;
- an **embedding** if F is full, faithful, and injective on objects;
- an **equivalence** if F is full, faithful, and *essentially surjective*: i.e. for every object B of \mathcal{D} there is an object A of \mathcal{C} such that $F(A) \cong B$;
- an **isomorphism** if there is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that

$$G \circ F = \text{Id}_{\mathcal{C}}, \quad F \circ G = \text{Id}_{\mathcal{D}}.$$

▲

We say that categories \mathcal{C} and \mathcal{D} are isomorphic, $\mathcal{C} \cong \mathcal{D}$, if there is an isomorphism between them. Note that this is just the usual notion of isomorphism applied to **Cat**. Examples:

- The forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$ is faithful, but not full. For the latter, note that not all functions $f : M \rightarrow N$ yield an arrow $f : (M, \cdot, 1) \rightarrow (N, \cdot, 1)$. Similar properties hold for other forgetful functors.
- The free monoid functor $\mathbf{MList} : \mathbf{Set} \rightarrow \mathbf{Mon}$ is faithful, but not full.
- The product functor $_ \times _ : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is generally neither faithful nor full. For the latter, e.g. in **Set**, the function $f : \mathbb{N}^2 \rightarrow \mathbb{N}^2 := (m, n) \mapsto (n, n)$ cannot be expressed in the form $f_1 \times f_2$. Faithfulness of the functor is examined in exercise 1.3.5(2).
- There is an equivalence between \mathbf{FDVect}_k the category of finite dimensional vector spaces over the field k , and \mathbf{Mat}_k , the category of matrices with entries in k . Note that these categories are very far from isomorphic! This example is elaborated in exercise 1.3.5(1).

Preservation and Reflection

Let P be a property of arrows. We say that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ *preserves* P if whenever f satisfies P , so does $F(f)$. We say that F *reflects* P if whenever $F(f)$ satisfies P , so does f . For example:

- All functors preserve isomorphisms, split monics and split epics.
 - Faithful functors reflect monics and epics.
 - Full and faithful functors reflect isomorphisms.
 - Equivalences preserve monics and epics.
- The forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$ preserves products.

Let us show c; the rest are given as exercises below. So let $f : A \rightarrow B$ in \mathcal{C} be such that Ff is an iso, that is, it has an inverse $g' : FB \rightarrow FA$. Then, by fullness, there exists some $g : B \rightarrow A$ so that $g' = Fg$. Thus,

$$F(g \circ f) = Fg \circ Ff = g' \circ Ff = \text{id}_{FA} = F(\text{id}_A).$$

By faithfulness we obtain $g \circ f = \text{id}_A$. Similarly, $f \circ g = \text{id}_B$ and therefore f is an isomorphism.

Exercise 49 Show items a, b and d above.

Exercise 50 Show the following.

- Functors do not in general reflect monics or epics.
- Faithful functors do not in general reflect isomorphisms.
- Full and faithful functors do not in general preserve monics or epics.

1.3.5 Exercises

1. Consider the category $\mathbf{FDVect}_{\mathbb{R}}$ of finite dimensional vector spaces over \mathbb{R} , and $\mathbf{Mat}_{\mathbb{R}}$ of matrices over \mathbb{R} . Concretely, $\mathbf{Mat}_{\mathbb{R}}$ is defined as follows:

$$\begin{aligned} \text{Ob}(\mathbf{Mat}_{\mathbb{R}}) &:= \mathbb{N}, \\ \mathbf{Mat}_{\mathbb{R}}(n, m) &:= \{M \mid M \text{ is an } n \times m \text{ matrix with entries in } \mathbb{R}\}. \end{aligned}$$

Thus, objects are natural numbers, and arrows $n \rightarrow m$ are $n \times m$ real matrices. Composition is matrix multiplication, and the identity on n is the $n \times n$ identity matrix.

Now let $F : \mathbf{Mat}_{\mathbb{R}} \rightarrow \mathbf{FDVect}_{\mathbb{R}}$ be the functor taking each n to the vector space \mathbb{R}^n and each $M : n \rightarrow m$ to the linear function

$$FM : \mathbb{R}^n \longrightarrow \mathbb{R}^m := (x_1, \dots, x_n) \mapsto [x_1, \dots, x_n]M$$

with the $1 \times m$ matrix $[x_1, \dots, x_n]M$ considered as a vector in \mathbb{R}^m . Show that F is full, faithful and essentially surjective, and hence that $\mathbf{FDVect}_{\mathbb{R}}$ and $\mathbf{Mat}_{\mathbb{R}}$ are equivalent categories. Are they isomorphic?

2. Let \mathcal{C} be a category with binary products such that, for each pair of objects A, B ,

$$\mathcal{C}(A, B) \neq \emptyset. \quad (*)$$

Show that the product functor $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is faithful.

Would F still be faithful in the absence of condition $(*)$?

1.4 Natural Transformations

“Categories were only introduced to allow functors to be defined; functors were only introduced to allow natural transformations to be defined.”

Just as categories have morphisms between them, namely functors, so functors have morphisms between them too — *natural transformations*.

1.4.1 Basics

Definition 51 Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A *natural transformation*

$$t : F \longrightarrow G$$

is a family of morphisms in \mathcal{D} indexed by objects A of \mathcal{C} ,

$$\{ t_A : FA \longrightarrow GA \}_{A \in \text{Ob}(\mathcal{C})}$$

such that, for all $f : A \rightarrow B$, the following diagram commutes.

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ t_A \downarrow & & \downarrow t_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

This condition is known as *naturality*.

If each t_A is an isomorphism, we say that t is a ***natural isomorphism***:

$$t : F \xrightarrow{\cong} G.$$

▲

Examples:

- Let Id be the identity functor on **Set**, and $\times \circ \langle \text{Id}, \text{Id} \rangle$ be the functor taking each set X to $X \times X$ and each function f to $f \times f$. Then, there is a natural transformation $\Delta : \text{Id} \rightarrow \times \circ \langle \text{Id}, \text{Id} \rangle$ given by:

$$\Delta_X : X \rightarrow X \times X := x \mapsto (x, x).$$

Naturality amounts to asserting that, for any function $f : X \rightarrow Y$, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

We call Δ the *diagonal* transformation on **Set**. In fact, it is the *only* natural transformation between these functors.

- The diagonal transformation can be defined for any category \mathcal{C} with binary products by setting, for each object A in \mathcal{C} ,

$$\Delta_A : A \rightarrow A \times A := \langle \text{id}_A, \text{id}_A \rangle.$$

Projections also yield natural transformations. For example the arrows

$$\pi_{1(A,B)} : A \times B \rightarrow A$$

specify a natural transformation $\pi_1 : \times \rightarrow \pi_1$. Note that $\times, \pi_1 : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ are the functors for product and first projection respectively.

- Let \mathcal{C} be a category with terminal object T , and let $K_T : \mathcal{C} \rightarrow \mathcal{C}$ be the functor mapping all objects to T and all arrows to id_T . Then, the canonical

arrows

$$\tau_A : A \longrightarrow T$$

specify a natural transformation $\tau : \mathbf{Id} \rightarrow K_T$ (where \mathbf{Id} the identity functor on \mathcal{C}).

- Recall the functor $\mathbf{List} : \mathbf{Set} \rightarrow \mathbf{Set}$ which takes a set X to the set of finite lists with elements in X . We can define (amongst others) the following natural transformations,

$$\text{reverse} : \mathbf{List} \longrightarrow \mathbf{List}, \quad \text{unit} : \mathbf{Id} \longrightarrow \mathbf{List}, \quad \text{flatten} : \mathbf{List} \circ \mathbf{List} \longrightarrow \mathbf{List},$$

by setting, for each set X ,

$$\text{reverse}_X : \mathbf{List}(X) \longrightarrow \mathbf{List}(X) := [x_1, \dots, x_n] \mapsto [x_n, \dots, x_1],$$

$$\text{unit}_X : X \longrightarrow \mathbf{List}(X) := x \mapsto [x],$$

$$\text{flatten}_X : \mathbf{List}(\mathbf{List}(X)) \longrightarrow \mathbf{List}(X)$$

$$:= [[x_1^1, \dots, x_{n_1}^1], \dots, [x_1^k, \dots, x_{n_k}^k]] \mapsto [x_1^1, \dots, x_{n_k}^k].$$

- Consider the functor $P := \times \circ \langle U, U \rangle$ with $U : \mathbf{Mon} \rightarrow \mathbf{Set}$, *i.e.*

$$P : \mathbf{Mon} \longrightarrow \mathbf{Set} := (M, \cdot, 1) \mapsto M \times M, \quad f \mapsto f \times f.$$

Then, the monoid operation yields a natural transformation $t : P \rightarrow U$ defined by:

$$t_{(M, \cdot, 1)} : M \times M \longrightarrow M := (m, m') \mapsto m \cdot m'.$$

Naturality corresponds to asserting that, for any $f : (M, \cdot, 1) \rightarrow (N, \cdot, 1)$, the following diagram commutes,

$$\begin{array}{ccc} M \times M & \xrightarrow{f \times f} & N \times N \\ \downarrow t_M & & \downarrow t_N \\ M & \xrightarrow{f} & N \end{array}$$

that is, for any $m_1, m_2 \in M$, $f(m_1) \cdot f(m_2) = f(m_1 \cdot m_2)$.

- If V is a finite dimensional vector space, then V is isomorphic to both its first dual V^* and to its second dual V^{**} .

However, while it is naturally isomorphic to its second dual, there is no natural isomorphism to the first dual. This was actually the original example which motivated Eilenberg and Mac Lane to define the concept of natural transformation; here naturality captures *basis independence*.

Exercise 52 Verify naturality of diagonal transformations, projections and terminals for a category \mathcal{C} with finite products.

Exercise 53 Prove that the diagonal is the only natural transformation $\text{Id} \longrightarrow \times \circ \langle \text{Id}, \text{Id} \rangle$ on **Set**. Similarly, prove that the first projection is the only natural transformation $\times \rightarrow \pi_1$ on **Set**.

1.4.2 Further Examples

Natural isomorphisms for products

Let \mathcal{C} be a category with finite products, *i.e.* binary products and a terminal object $\mathbf{1}$. Then, we have the following canonical natural isomorphisms.

$$\begin{aligned} a_{A,B,C} : A \times (B \times C) &\xrightarrow{\cong} (A \times B) \times C, \\ s_{A,B} : A \times B &\xrightarrow{\cong} B \times A, \\ l_A : \mathbf{1} \times A &\xrightarrow{\cong} A, \\ r_A : A \times \mathbf{1} &\xrightarrow{\cong} A. \end{aligned}$$

The first two isomorphisms are meant to assert that the product is *associative* and *symmetric*, and the last two that $\mathbf{1}$ is its *unit*. In later sections we will see that these conditions form part of the definition of *symmetric monoidal categories*.

These natural isomorphisms are defined explicitly by:

$$\begin{aligned} a_{A,B,C} &:= \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle, \\ s_{A,B} &:= \langle \pi_2, \pi_1 \rangle, \\ l_A &:= \pi_2, \\ r_A &:= \pi_1. \end{aligned}$$

Since natural isomorphisms are a *self-dual* notion, similar natural isomorphisms can be defined if \mathcal{C} has binary coproducts and an initial object.

Exercise 54 Verify that these families of arrows are natural isomorphisms.

Natural transformations between Hom-functors

Let $f : A \rightarrow B$ in a category \mathcal{C} . Then, this induces a natural transformation

$$\begin{aligned}\mathcal{C}(f, _): \mathcal{C}(B, _) &\longrightarrow \mathcal{C}(A, _), \\ \mathcal{C}(f, _)_C: \mathcal{C}(B, C) &\longrightarrow \mathcal{C}(A, C) := (g: B \rightarrow C) \mapsto (g \circ f: A \rightarrow C).\end{aligned}$$

Note that $\mathcal{C}(f, _)_C$ is the same as $\mathcal{C}(f, C)$, the result of applying the contravariant functor $\mathcal{C}(_, C)$ to f . Hence, naturality amounts to asserting that, for each $h: C \rightarrow D$, the following diagram commutes.

$$\begin{array}{ccc}\mathcal{C}(B, C) & \xrightarrow{\mathcal{C}(B, h)} & \mathcal{C}(B, D) \\ \mathcal{C}(f, C) \downarrow & & \downarrow \mathcal{C}(f, D) \\ \mathcal{C}(A, C) & \xrightarrow{\mathcal{C}(A, h)} & \mathcal{C}(A, D)\end{array}$$

Starting from a $g: B \rightarrow C$, we compute:

$$\mathcal{C}(A, h)(\mathcal{C}(f, C)(g)) = h \circ (g \circ f) = (h \circ g) \circ f = \mathcal{C}(f, D)(\mathcal{C}(B, h)(g)).$$

The natural transformation $\mathcal{C}(_, f): \mathcal{C}(_, A) \rightarrow \mathcal{C}(_, B)$ is defined similarly.

Exercise 55 Define the natural transformation $\mathcal{C}(_, f)$ and verify its naturality.

There is a remarkable result, the **Yoneda Lemma**, which says that *every* natural transformation between Hom-functors comes from a (unique) arrow in \mathcal{C} in the fashion described above.

Lemma 1. *Let A, B be objects in a category \mathcal{C} . For each natural transformation $t: \mathcal{C}(A, _) \rightarrow \mathcal{C}(B, _)$, there is a unique arrow $f: B \rightarrow A$ such that*

$$t = \mathcal{C}(f, _).$$

Proof: Take any such A, B and t and let

$$f: B \longrightarrow A := t_A(\text{id}_A).$$

We want to show that $t = \mathcal{C}(f, _)$. For any object C and any arrow $g: A \rightarrow C$, naturality of t means that the following commutes.

$$\begin{array}{ccc}\mathcal{C}(A, A) & \xrightarrow{\mathcal{C}(A, g)} & \mathcal{C}(A, C) \\ t_A \downarrow & & \downarrow t_C \\ \mathcal{C}(B, A) & \xrightarrow{\mathcal{C}(B, g)} & \mathcal{C}(B, C)\end{array}$$

Starting from id_A we have that:

$$t_C(\mathcal{C}(A, g)(\text{id}_A)) = \mathcal{C}(B, g)(t_A(\text{id}_A)), \quad \text{i.e. } t_C(g) = g \circ f.$$

Hence, noting that $\mathcal{C}(f, C)(g) = g \circ f$, we obtain $t = \mathcal{C}(f, -)$.

For uniqueness we have that, for any $f, f' : B \rightarrow A$, if $\mathcal{C}(f, -) = \mathcal{C}(f', -)$ then

$$f = \text{id}_A \circ f = \mathcal{C}(f, A)(\text{id}_A) = \mathcal{C}(f', A)(\text{id}_A) = \text{id}_A \circ f' = f'.$$

■

Exercise 56 Prove a similar result for contravariant hom-functors.

Alternative definition of equivalence

Another way of defining equivalence of categories is as follows.

Definition 57 We say that categories \mathcal{C} and \mathcal{D} are *equivalent*, $\mathcal{C} \simeq \mathcal{D}$, if there are functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms

$$G \circ F \cong \text{id}_{\mathcal{C}}, \quad F \circ G \cong \text{id}_{\mathcal{D}}.$$

▲

1.4.3 Functor Categories

Suppose we have functors $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ and natural transformations

$$t : F \longrightarrow G, \quad u : G \longrightarrow H.$$

Then, we can compose these natural transformations, yielding $u \circ t : F \rightarrow H$:

$$(u \circ t)_A := FA \xrightarrow{t_A} GA \xrightarrow{u_A} HA.$$

Composition is associative, and has as identity the natural transformation

$$I_F : F \longrightarrow F := \{ (I_F)_A := \text{id}_A : FA \longrightarrow FA \}_A.$$

These observations lead us to the following.

Definition 58 For categories \mathcal{C}, \mathcal{D} define the *functor category* $\text{Func}(\mathcal{C}, \mathcal{D})$ by taking:

- Objects: functors $F : \mathcal{C} \rightarrow \mathcal{D}$.
- Arrows: natural transformations $t : F \rightarrow G$.

Composition and identities are given as above.

▲

Remark 59 We see that in the category **Cat** of categories and functors, each hom-set $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ *itself has the structure of a category*. In fact, **Cat** is the basic example of a “2-category”, *i.e.* of a category where hom-sets are themselves categories.

Note that a natural isomorphism is precisely an isomorphism in the functor category. Let us proceed to some examples of functor categories.

- Recall that, for any group G , functors from G to **Set** are G -actions on sets. Then, $\mathbf{Func}(G, \mathbf{Set})$ is the category of G -actions on sets and *equivariant functions*: $f : X \rightarrow Y$ such that $f(m \bullet x) = m \bullet f(x)$.
- $\mathbf{Func}(2_{\Rightarrow}, \mathbf{Set})$: Graphs and graph homomorphisms.
- If $F, G : P \rightarrow Q$ are monotone maps between posets, then $t : F \rightarrow G$ means that

$$\forall x \in P. Fx \leq Gx.$$

Note that in this case naturality is trivial (hom-sets are singletons in Q).

Exercise 60 Verify the above descriptions of $\mathbf{Func}(G, \mathbf{Set})$ and $\mathbf{Func}(2_{\Rightarrow}, \mathbf{Set})$.

Remark 61 The composition of natural transformations defined above is called *vertical composition*. The reason for this terminology is depicted below.

As expected, there is also a *horizontal composition*, which is given as follows.

1.4.4 Exercises

1. By identifying the relevant functors, express pairing $\langle _, _ \rangle$ as a natural transformation. What does naturality correspond to explicitly?
2. Show that the two definitions of equivalence of categories, namely
 - a. \mathcal{C} and \mathcal{D} are equivalent if there is an equivalence $F : \mathcal{C} \rightarrow \mathcal{D}$ (definition 48),

- b. \mathcal{C} and \mathcal{D} are equivalent if there are $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$, and isomorphisms $F \circ G \cong \text{Id}_{\mathcal{D}}$, $G \circ F \cong \text{Id}_{\mathcal{C}}$ (definition 57),
are: equivalent! Note that this will need the Axiom of Choice.
3. Define a relation on objects in a category \mathcal{C} by: $A \cong B$ iff A and B are isomorphic.
- a. Show that this relation is an equivalence relation.
- Define a *skeleton* of \mathcal{C} to be the (full) subcategory obtained by choosing one object from each equivalence class of \cong (note that this involves choices, and is not uniquely defined).
- b. Show that \mathcal{C} is equivalent to any skeleton.
- c. Show that any two skeletons of \mathcal{C} are isomorphic.
- d. Give an example of a category whose objects form a proper class, but whose skeleton is finite.
4. Given a category \mathcal{C} , we can define a functor

$$y : \mathcal{C} \longrightarrow \text{Func}(\mathcal{C}^{\text{op}}, \mathbf{Set}) := A \mapsto \mathcal{C}(-, A), \quad f \mapsto \mathcal{C}(-, f).$$

- Prove carefully that this is indeed a functor. Use exercise 56 to conclude that y is full and faithful. Prove that it is also injective on objects, and hence an embedding. It is known as the *Yoneda embedding*.
5. Define the horizontal composition $u \bullet t$ of natural transformations explicitly. Prove that it is associative.

1.5 Universality and Adjoints

There is a fundamental triad of categorical notions:

Functoriality, Naturality, Universality.

We have studied the first two notions explicitly. We have also seen many *examples* of universal definitions, notably the various notions of limits and colimits considered in section 1.2. It is now time to consider universality in general; the proper formulation of this fundamental and pervasive notion is one of the major achievements of basic category theory.

Universality arises when we are interested in finding *canonical solutions* to problems of construction: that is, we are interested not just in the *existence* of a solution but in its *canonicity*. This canonicity should guarantee uniqueness, in the sense we have become familiar with: a canonical solution should be *unique up to (unique) isomorphism*.

The notion of canonicity has a simple interpretation in the case of posets, as an *extremal solution*: one that is the least or the greatest among all solu-

tions. Such an extremal solution is obviously unique. For example, consider the problem of finding a lower bound of a pair of elements A, B in a poset P : a *greatest lower bound* of A and B is an extremal solution to this problem. As we have seen, this is the specialisation to posets of the problem of constructing a product:

- \rightsquigarrow A product of A, B in a poset is an element C such that $C \leq A$ and $C \leq B$, (C is a lower bound);
- \rightsquigarrow and for any other solution C' , *i.e.* C' such that $C' \leq A$ and $C' \leq B$, we have $C' \leq C$. (C is a greatest lower bound.)

Because the ideas of universality and adjunctions have an appealingly simple form in posets, which is, moreover, useful in its own right, we will develop the ideas in that special case first, as a prelude to the general discussion for categories.

1.5.1 Adjunctions for Posets

Suppose $g : Q \rightarrow P$ is a monotone map between posets. Given $x \in P$, a ***g -approximation of x*** (from above) is an element $y \in Q$ such that $x \leq g(y)$. A ***best g -approximation of x*** is an element $y \in Q$ such that

$$x \leq g(y) \wedge \forall z \in Q. (x \leq g(z) \implies y \leq z).$$

If a best g -approximation exists then it is clearly unique.

Discussion

It is worth clarifying the notion of best g -approximation. If y is a best g -approximation to x , then in particular, by monotonicity of g , $g(y)$ is the least element of the set of all $g(z)$ where $z \in Q$ and $x \leq g(z)$. However, the property of being a best approximation is much *stronger* than the mere existence of a least element of this set. We are asking for y *itself* to be the least, *in Q* , among all elements z such that $x \leq g(z)$. Thus, even if g is *surjective*, so that for every x there is a $y \in Q$ such that $g(y) = x$, there need not exist a *best* g -approximation to x . This is exactly the issue of having a *canonical choice* of solution.

Exercise 62 Give an example of a surjective monotone map $g : Q \rightarrow P$ and an element $x \in P$ such that there is no best g -approximation to x in Q .

If such a best g -approximation $f(x)$ exists for all $x \in P$ then we have a function $f : P \rightarrow Q$ such that, for all $x \in P, z \in Q$:

$$x \leq g(z) \iff f(x) \leq z. \quad (1.1)$$

We say that f is the **left adjoint** of g , and g is the **right adjoint** of f . It is immediate from the definitions that the left adjoint of g , if it exists, is uniquely determined by g .

Proposition 63 *If such a function f exists, then it is monotone. Moreover,*

$$\text{id}_P \leq g \circ f, \quad f \circ g \leq \text{id}_Q, \quad f \circ g \circ f = f, \quad g \circ f \circ g = g.$$

Proof: If we take $z = f(x)$ in equation (1.1), then since $f(x) \leq f(x)$, $x \leq g \circ f(x)$. Similarly, taking $x = g(z)$ we obtain $f \circ g(z) \leq z$. Now, the ordering on functions $h, k : P \rightarrow Q$ is the *pointwise order*:

$$h \leq k \iff \forall x \in P. h(x) \leq k(x).$$

This gives the first two equations.

Now, if $x \leq_P x'$ then $x \leq x' \leq g \circ f(x')$, so $f(x')$ is a g -approximation of x , and hence $f(x) \leq f(x')$. Thus, f is monotone.

Finally, using the fact that composition is monotone with respect to the pointwise order on functions, and the first two equations:

$$g = \text{id}_P \circ g \leq g \circ f \circ g \leq g \circ \text{id}_Q = g,$$

and hence $g = g \circ f \circ g$. The other equation is proved similarly. ■

Examples:

- Consider the inclusion map

$$i : \mathbb{Z} \hookrightarrow \mathbb{R}.$$

This has both a left adjoint f^L and a right adjoint f^R , where $f^L, f^R : \mathbb{R} \rightarrow \mathbb{Z}$. For all $z \in \mathbb{Z}$, $r \in \mathbb{R}$:

$$z \leq f^R(r) \iff i(z) \leq r, \quad f^L(r) \leq z \iff r \leq i(z).$$

We see from these defining properties that the right adjoint maps a real r to the *greatest integer below it* (the extremal solution to finding an integer below a given real). This is the standard *floor function*.

Similarly, the left adjoint maps a real to the least integer above it yielding the *ceiling function*. Thus:

$$f^R(r) = \lfloor r \rfloor, \quad f^L(r) = \lceil r \rceil.$$

- Consider a relation $R \subseteq X \times Y$. R induces a function:

$$f_R : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) := S \mapsto \{y \in Y \mid \exists x \in S. xRy\}.$$

This has a right adjoint $[R] : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$:

$$S \subseteq [R]T \iff f_R(S) \subseteq T.$$

The definition of $[R]$ which satisfies this condition is:

$$[R]T := \{x \in X \mid \forall y \in Y. xRy \Rightarrow y \in T\}.$$

If we consider a set of *worlds* W with an *accessibility relation* $R \subseteq W \times W$ as in Kripke semantics for modal logic, we see that $[R]$ gives the usual Kripke semantics for the modal operator \Box , seen as a propositional operator mapping the set of worlds satisfied by a formula ϕ to the set of worlds satisfied by $\Box\phi$.

On the other hand, if we think of the relation R as the denotation of a (possibly non-deterministic) program, and T as a predicate on *states*, then $[R]T$ is exactly the *weakest precondition* $\mathbf{wp}(R, T)$. In *Dynamic Logic*, the two settings are combined, and we can write expressions such as $[R]T$ directly, where T will be (the denotation of) some formula, and R the relation corresponding to a program.

- Consider a function $f : X \rightarrow Y$. This induces a function:

$$f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X) := T \mapsto \{x \in X \mid f(x) \in T\}.$$

This function f^{-1} has both a left adjoint $\exists(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, and a right adjoint $\forall(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$. For all $S \subseteq X$, $T \subseteq Y$:

$$\exists(f)(S) \subseteq T \iff S \subseteq f^{-1}(T), \quad f^{-1}(T) \subseteq S \iff T \subseteq \forall(f)(S).$$

How can we define $\forall(f)$ and $\exists(f)$ explicitly so as to fulfil these defining conditions? – As follows:

$$\begin{aligned} \exists(f)(S) &:= \{y \in Y \mid \exists x \in X. f(x) = y \wedge x \in S\}, \\ \forall(f)(S) &:= \{y \in Y \mid \forall x \in X. f(x) = y \Rightarrow x \in S\}. \end{aligned}$$

If $R \subseteq X \times Y$, which we write in logical notation as $R(x, y)$, and we take the projection function $\pi_1 : X \times Y \longrightarrow X$, then:

$$\forall(\pi_1)(R) \equiv \forall y. R(x, y), \quad \exists(\pi_1)(R) \equiv \exists y. R(x, y).$$

This extends to an algebraic form of the usual Tarski model-theoretic semantics for first-order logic, in which:

Quantifiers are Adjoints

Couniversality

We can dualise the discussion, so that starting with a monotone map $f : P \rightarrow Q$ and $y \in Q$, we can ask for the best P -approximation to y from below: $x \in P$ such that $f(x) \leq y$, and for all $z \in P$:

$$f(z) \leq y \iff z \leq x.$$

If such a best approximation $g(y)$ exists for all $y \in Q$, we obtain a monotone map $g : Q \rightarrow P$ such that g is right adjoint to f . From the symmetry of the definition, it is clear that:

$$f \text{ is the left adjoint of } g \iff g \text{ is the right adjoint of } f$$

and each determines the other uniquely.

1.5.2 Universal Arrows and Adjoints

Our discussion of best approximations for posets is lifted to general categories as follows.

Definition 64 Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor, and C an object of \mathcal{C} . A **universal arrow from C to G** is a pair (D, η) where D is an object of \mathcal{D} and

$$\eta : C \longrightarrow G(D),$$

such that, for any object D' of \mathcal{D} and morphism $f : C \rightarrow G(D')$, there exists a unique morphism $\hat{f} : D \rightarrow D'$ in \mathcal{D} such that $f = G(\hat{f}) \circ \eta$.

Diagrammatically:

$$\begin{array}{ccc} C & \xrightarrow{\eta} & G(D) \\ & \searrow f & \downarrow G(\hat{f}) \\ & & G(D') \end{array} \qquad \begin{array}{c} D \\ \downarrow \hat{f} \\ D' \end{array} \quad \blacktriangle$$

As in previous cases, uniqueness can be given a purely equational specification:

$$\forall h : D \longrightarrow D'. \widehat{G(h) \circ \eta} = h. \quad (1.2)$$

Exercise 65 Show that if (D, η) and (D', η') are universal arrows from C to G then there is a unique isomorphism $D \cong D'$.

Exercise 66 Check that the equational specification of uniqueness (1.2) is valid.

Examples:

- Take $U : \mathbf{Mon} \rightarrow \mathbf{Set}$. Given a set X , the universal arrow is

$$\eta_X : X \longrightarrow U(\mathbf{MList}(X)) := x \mapsto [x].$$

Indeed, for any monoid $(M, \cdot, 1)$ and any function $f : X \rightarrow M$, set

$$\hat{f} : \mathbf{MList}(X) \longrightarrow (M, \cdot, 1) := [x_1, \dots, x_n] \mapsto f(x_1) \cdot \dots \cdot f(x_n).$$

It is easy to see that \hat{f} is a monoid homomorphism, and that $U(\hat{f}) \circ \eta_X = f$. Moreover, for uniqueness we have that, for any $h : \mathbf{MList}(X) \rightarrow (M, \cdot, 1)$,

$$\begin{aligned} U(\widehat{h \circ \eta_X}) &= x \mapsto \widehat{h([x])} = [x_1, \dots, x_n] \mapsto h([x_1]) \cdot \dots \cdot h([x_n]) \\ &= [x_1, \dots, x_n] \mapsto h([x_1]) * \dots * h([x_n]) \\ &= [x_1, \dots, x_n] \mapsto h([x_1, \dots, x_n]) = h. \end{aligned}$$

- Let $K : \mathcal{C} \rightarrow \mathbb{1}$ be the unique functor to the one-object/one-arrow category. A universal arrow from the object of $\mathbb{1}$ to K corresponds to an initial object in \mathcal{C} .
Indeed, such a universal arrow is given by an object I of \mathcal{C} (and a trivial arrow in $\mathbb{1}$), such that for any A in \mathcal{C} (and relevant arrow in $\mathbb{1}$) there exists a unique arrow from I to A (such that a trivial condition holds).
- Consider the functor $\langle \text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}} \rangle : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$, taking each object A to (A, A) and each arrow f to (f, f) . A universal arrow from an object (A, B) of $\mathcal{C} \times \mathcal{C}$ to $\langle \text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}} \rangle$ corresponds to a coproduct of A and B .

Exercise 67 Verify the description of coproducts as universal arrows.

As in the case of posets, a related notion to universal arrows is that of *adjunction*.

Definition 68 Let \mathcal{C}, \mathcal{D} be categories. An **adjunction** from \mathcal{C} to \mathcal{D} is a triple (F, G, θ) , where F and G are functors

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

and θ is a family of bijections

$$\theta_{A,B} : \mathcal{C}(A, G(B)) \xrightarrow{\cong} \mathcal{D}(F(A), B),$$

for each $A \in \text{Ob}(\mathcal{C})$ and $B \in \text{Ob}(\mathcal{D})$, **natural** in A and B .

We say that F is **left adjoint** to G , and G is **right adjoint** to F . ▲

Note that θ should be understood as the “witnessed” form — *i.e.* arrows instead of mere relations — of the defining condition for adjunctions in the case of posets:

$$x \leq g(y) \iff f(x) \leq y.$$

This is often displayed as a two-way ‘inference rule’:

$$\frac{A \longrightarrow GB}{FA \longrightarrow B}$$

Naturality of θ is expressed as follows: for any $f : A \rightarrow G(B)$ and any $g : A' \rightarrow A$, $h : B \rightarrow B'$,

$$\begin{aligned} \theta_{A',B}(f \circ g) &= \theta_{A,B}(f) \circ F(g), \\ \theta_{A,B'}(G(h) \circ f) &= h \circ \theta_{A,B}(f). \end{aligned}$$

Note that f, g are in \mathcal{C} , and h is in \mathcal{D} . In one line:

$$\theta_{A',B'}(G(h) \circ f \circ g) = h \circ \theta_{A,B}(f) \circ F(g).$$

Diagrammatically:

$$\begin{array}{ccccc} \mathcal{C}(A, GB') & \xleftarrow{\mathcal{C}(A, Gh)} & \mathcal{C}(A, GB) & \xrightarrow{\mathcal{C}(g, GB)} & \mathcal{C}(A', GB) & \mathcal{C}(A, GB) & \xrightarrow{\mathcal{C}(g, Gh)} & \mathcal{C}(A', GB') \\ \theta_{A,B'} \downarrow & & \theta_{A,B} \downarrow & & \theta_{A',B} \downarrow & \theta_{A,B} \downarrow & & \theta_{A',B'} \downarrow \\ \mathcal{D}(FA, B') & \xleftarrow{\mathcal{D}(FA, h)} & \mathcal{D}(FA, B) & \xrightarrow{\mathcal{D}(Fg, B)} & \mathcal{D}(FA', B) & \mathcal{D}(FA, B) & \xrightarrow{\mathcal{D}(Fg, h)} & \mathcal{D}(FA', B') \end{array}$$

Thus, θ is in fact a natural isomorphism

$$\theta : \mathcal{C}(_, G(_)) \xrightarrow{\cong} \mathcal{D}(F(_), _),$$

where $\mathcal{C}(_, G(_)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$ is the result of composing the bivariate hom-functor $\mathcal{C}(_, _)$ with $\text{Id}_{\mathcal{C}^{\text{op}}} \times G$, and $\mathcal{D}(F(_), _)$ is similar.

In the next propositions we show that universal arrows and adjunctions are equivalent notions.

Proposition 69 (Universals define adjunctions) *Let $G : \mathcal{D} \rightarrow \mathcal{C}$. If for every object C of \mathcal{C} there exists a universal arrow $\eta_C : C \rightarrow G(F(C))$, then:*

1. F uniquely extends to a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $\eta : \text{Id}_{\mathcal{C}} \rightarrow G \circ F$ is a natural transformation.
2. F is uniquely determined by G (up to unique natural isomorphism), and vice versa.
3. For each pair of objects C of \mathcal{C} and D of \mathcal{D} , there is a natural bijection:

$$\theta_{C,D} : \mathcal{C}(C, G(D)) \cong \mathcal{D}(F(C), D).$$

Proof: For 1, we extend F to a functor as follows. Given $f : C \rightarrow C'$ in \mathcal{C} , we consider the composition

$$\eta_{C'} \circ f : C \longrightarrow GFC'.$$

By the universal property of η_C , there exists a unique arrow $Ff : FC \rightarrow FC'$ such that the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & GFC \\ f \downarrow & & \downarrow GFf \\ C' & \xrightarrow{\eta_{C'}} & GFC' \end{array}$$

Note that the above is the naturality diagram for η on C , hence the arrow-map thus defined for F is the unique candidate that makes η a natural transformation.

It remains to verify the functoriality of F . To show that F preserves composition, consider $g : C' \rightarrow C''$. We have the following commutative diagram,

$$\begin{array}{ccccc} C & \xrightarrow{f} & C' & \xrightarrow{g} & C'' \\ \eta_C \downarrow & & \eta_{C'} \downarrow & & \downarrow \eta_{C''} \\ GFC & \xrightarrow{GFf} & GFC' & \xrightarrow{GFg} & GFC'' \end{array}$$

from which it follows that

$$G(Fg \circ Ff) \circ \eta_C = GFg \circ GFf \circ \eta_C = \eta_{C''} \circ g \circ f,$$

$$\therefore F(g \circ f) = \widehat{\eta_{C''} \circ g \circ f} = G(\widehat{Fg \circ Ff}) \circ \eta_C = Fg \circ Ff,$$

where the last equality above holds because of (1.2). The verification that F preserves identities is similar.

For 2, we have that each FC is determined uniquely up to unique isomorphism, by the universal property, and once the object part of F is fixed, the arrow part is uniquely determined.

For 3, we need to define a natural isomorphism $\theta_{C,D} : \mathcal{C}(C, G(D)) \cong \mathcal{D}(F(C), D)$. Given $f : C \rightarrow GD$, $\theta_{C,D}(f)$ is defined to be the unique arrow $FC \rightarrow D$ such that the following commutes, as dictated by universality.

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & GFC \\ & \searrow f & \downarrow G(\theta_{C,D}(f)) \\ & & GD \end{array}$$

Suppose that $\theta_{C,D}(f) = \theta_{C,D}(g)$. Then

$$f = G(\theta_{C,D}(f)) \circ \eta_C = G(\theta_{C,D}(g)) \circ \eta_C = g.$$

Thus $\theta_{C,D}$ is injective. Moreover, given $h : FC \rightarrow D$, by the equational formulation of uniqueness (1.2) we have:

$$h = \theta_{C,D}(Gh \circ \eta_C).$$

Thus $\theta_{C,D}$ is surjective. We are left to show naturality, *i.e.* that the following diagram commutes, for all $h : C' \rightarrow C$ and $g : D \rightarrow D'$.

$$\begin{array}{ccc} \mathcal{C}(C, GD) & \xrightarrow{\mathcal{C}(h, Gg)} & \mathcal{C}(C', GD') \\ \theta_{C,D} \downarrow & & \downarrow \theta_{C',D'} \\ \mathcal{D}(FC, D) & \xrightarrow{\mathcal{D}(Fh, g)} & \mathcal{D}(FC', D') \end{array}$$

We chase around the diagram, starting from $f : C \rightarrow GD$.

$$\mathcal{D}(Fh, g) \circ \theta_{C,D}(f) = g \circ \theta_{C,D}(f) \circ Fh$$

$$\theta_{C',D'} \circ \mathcal{C}(h, Gg)(f) = \theta_{C',D'}(Gg \circ f \circ h)$$

Now:

$$\begin{aligned} g \circ \theta_{C,D}(f) \circ Fh &= \theta_{C',D'}(G(g \circ \theta_{C,D}(f) \circ Fh) \circ \eta_{C'}) && \text{by (1.2)} \\ &= \theta_{C',D'}(Gg \circ G(\theta_{C,D}(f)) \circ GFh \circ \eta_{C'}) && \text{functoriality of } G \\ &= \theta_{C',D'}(Gg \circ G(\theta_{C,D}(f)) \circ \eta_C \circ h) && \text{naturality of } \eta \\ &= \theta_{C',D'}(Gg \circ f \circ h) && \text{by (1.2)}. \end{aligned}$$

■

Proposition 70 (Adjunctions define universals) *Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor, $D \in \text{Ob}(\mathcal{D})$ and $C \in \text{Ob}(\mathcal{C})$. If, for any $D' \in \text{Ob}(\mathcal{D})$, there is a bijection*

$$\phi_{D'} : \mathcal{C}(C, G(D')) \cong \mathcal{D}(D, D')$$

natural in D' then there is a universal arrow $\eta : C \rightarrow G(D)$.

Proof: Take $\eta : C \rightarrow G(D) := \phi_D^{-1}(\text{id}_D)$ and, for any $g : C \rightarrow G(D')$, take $\hat{g} : D \rightarrow D' := \phi_{D'}(g)$.

We have that

$$G(\hat{g}) \circ \eta = G(\hat{g}) \circ \phi_D^{-1}(\text{id}_D) \stackrel{\text{nat}}{=} \phi_{D'}^{-1}(\hat{g}) = g.$$

Moreover, for any $h : D \rightarrow D'$,

$$\phi_{D'}(Gh \circ \eta) = \phi_{D'}(Gh \circ \phi_D^{-1}(\text{id}_D)) \stackrel{\text{nat}}{=} \phi_{D'}(\phi_{D'}^{-1}(h)) = h,$$

where equalities labelled with “nat” hold because of naturality of ϕ . ■

Corollary 71 *Let (F, G, θ) be an adjunction with $F : \mathcal{C} \rightarrow \mathcal{D}$. Then, for each $C \in \text{Ob}(\mathcal{C})$ there is a universal arrow $\eta : C \rightarrow G(F(C))$.* ■

Equivalence of Universals and Adjoints

Thus we see that the following two situations are equivalent, in the sense that each determines the other uniquely.

- We are given a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, and for each object C of \mathcal{C} a universal arrow from C to G .
- We are given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$, and a natural bijection

$$\theta_{C,D} : \mathcal{C}(C, G(D)) \cong \mathcal{D}(F(C), D).$$

Couniversal Arrows

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and D an object of \mathcal{D} . A **couniversal arrow from F to D** is an object C of \mathcal{C} and a morphism

$$\epsilon : F(C) \longrightarrow D$$

such that, for every object C' of \mathcal{C} and morphism $g : F(C') \rightarrow D$, there exists a unique morphism $\bar{g} : C' \rightarrow C$ in \mathcal{C} such that $g = \epsilon \circ F(\bar{g})$.

Diagrammatically:

$$\begin{array}{ccc} C & & F(C) \xrightarrow{\epsilon} D \\ \uparrow & & \uparrow \\ \bar{g} \downarrow & & F(\bar{g}) \downarrow \\ C' & & F(C') \end{array} \quad \begin{array}{c} \nearrow g \end{array}$$

By exactly similar (but dual) reasoning to the previous propositions, an adjunction implies the existence of couniversal arrows, and the existence of the latter implies the existence of the adjunction. Hence,

$$\text{Universality} \equiv \text{Adjunctions} \equiv \text{Couniversality}.$$

Some examples of couniversal arrows:

- A terminal object in a category \mathcal{C} is a couniversal arrow from the unique functor $K : \mathcal{C} \rightarrow \mathbf{1}$ to the unique object in $\mathbf{1}$.

- Let A, B be objects of \mathcal{C} . A product of A and B is a couniversal arrow from $\langle \text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}} \rangle : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ to (A, B) .

1.5.3 Limits and Colimits

In the previous paragraph we described products $A \times B$ as couniversal arrows from the *diagonal functor* $\Delta : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ to (A, B) . Δ is the functor assigning (A, A) to each object A , and (f, f) to each arrow f . Noting that $\mathcal{C} \times \mathcal{C} = \mathcal{C}^2$, where \mathcal{C}^2 is a *functor category*, this suggests an important generalisation.

Definition 72 Let \mathcal{C} be a category and \mathcal{I} be another category, thought of as an “index category”. A *diagram of shape \mathcal{I}* in \mathcal{C} is just a functor $F : \mathcal{I} \rightarrow \mathcal{C}$. Consider the functor category $\mathcal{C}^{\mathcal{I}}$ with objects the functors from \mathcal{I} to \mathcal{C} , and natural transformations as morphisms. There is a *diagonal functor*

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{I}},$$

taking each object C of \mathcal{C} to the constant functor $K_C : \mathcal{I} \rightarrow \mathcal{C}$, which maps every object of \mathcal{I} to C . A **limit** for the diagram F is a couniversal arrow from Δ to F . ▲

This concept of limit subsumes products (including infinite products), pullbacks, inverse limits, etc.

For example, take $\mathcal{I} := 2_{\Rightarrow}$ (we have seen this before: $2_{\Rightarrow} = \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet$). A functor F from \mathcal{I} to \mathcal{C} corresponds to a diagram:

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} B$$

A couniversal arrow from Δ to F corresponds to the following situation,

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} & B \\ \uparrow \hat{h} & \nearrow h & & & \\ C & & & & \end{array}$$

i.e. to an equaliser!

By dualising limits we obtain *colimits*. Some important examples are co-products, coequalisers, pushouts and ω -colimits.

Exercise 73 Verify that pullbacks are limits by taking:

$$\mathcal{I} := \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Limits as terminal objects

Consider $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$ and $F : \mathcal{I} \rightarrow \mathcal{C}$. A *cone to F* is an object C of \mathcal{C} and family of arrows γ ,

$$\{ \gamma_I : C \longrightarrow FI \}_{I \in \text{Ob}(\mathcal{I})},$$

such that, for any $f : I \rightarrow J$, the following triangle commutes.

$$\begin{array}{ccc} FI & \xrightarrow{Ff} & FJ \\ & \swarrow \gamma_I \quad \searrow \gamma_J & \\ & C & \end{array}$$

Thus a cone is exactly a natural transformation $\gamma : \Delta C \rightarrow F$. A morphism of cones (‘mediating morphism’) $(C, \gamma) \rightarrow (D, \delta)$ is an arrow $g : C \rightarrow D$ such that each of the following triangles commutes.

$$\begin{array}{ccc} & FI & \\ \gamma_I \nearrow & & \nwarrow \delta_I \\ C & \xrightarrow{g} & D \end{array}$$

We obtain a category **Cone**(F) whose objects are cones to F and whose arrows are mediating morphisms. Then, a limit of F is a terminal object in **Cone**(F).

1.5.4 Exponentials

In **Set**, given sets A, B , we can form the set of functions $B^A := \mathbf{Set}(A, B)$, which is again a set, i.e. an object of **Set**. This closure of **Set** under forming “function spaces” is one of its most important properties.

How can we axiomatise this situation? Once again, rather than asking what the elements of a function space *are*, we ask instead what we can *do* with them operationally. The answer is simple: apply functions to their arguments. That is, there is a map

$$\text{ev}_{A,B} : B^A \times A \longrightarrow B \quad \text{such that} \quad \text{ev}_{A,B}(f, a) = f(a).$$

We can think of the function as a ‘black box’: we can feed it inputs and observe the outputs.

Evaluation has the following couniversal property. For any $g : C \times A \rightarrow B$, there is a unique map $\Lambda(g) : C \rightarrow B^A$ such that the following diagram commutes.

$$\begin{array}{ccc}
 B^A \times A & \xrightarrow{\text{ev}_{A,B}} & B \\
 \uparrow \Lambda(g) \times \text{id}_A & \nearrow g & \\
 C \times A & &
 \end{array}$$

In **Set**, this is defined by:

$$\Lambda(g)(c) : A \longrightarrow B := a \mapsto g(c, a).$$

This process of transforming a function of two arguments into a function-valued function of one argument is known as *currying*, after H. B. Curry. It is an algebraic form of *λ -abstraction*.

We are now led to the general definition of exponentials. Note that, for each object A of a category \mathcal{C} with products, we can define a functor

$$_ \times A : \mathcal{C} \longrightarrow \mathcal{C}.$$

Definition 74 Let \mathcal{C} be a category with binary products. We say that \mathcal{C} **has exponentials** if for all objects A and B of \mathcal{C} there is a couniversal arrow from $_ \times A$ to B , *i.e.* an object B^A of \mathcal{C} and a morphism

$$\text{ev}_{A,B} : B^A \times A \longrightarrow B$$

with the couniversal property: for every $g : C \times A \rightarrow B$, there is a unique morphism $\Lambda(g) : C \rightarrow B^A$ such that the following diagram commutes.

$$\begin{array}{ccc}
 B^A \times A & \xrightarrow{\text{ev}_{A,B}} & B \\
 \uparrow \Lambda(g) \times \text{id}_A & \nearrow g & \\
 C \times A & &
 \end{array}$$

▲

As before, the couniversal property can be given in purely equational terms, as follows. For every $h : C \rightarrow B^A$,

$$\Lambda(\text{ev}_{A,B} \circ h \times \text{id}_A) = h.$$

Equivalently, \mathcal{C} has exponentials if, for every object A , the functor $_ \times A$ has a right adjoint, that is, there exists a functor $_^A : \mathcal{C} \rightarrow \mathcal{C}$ and a bijection

$$\Lambda_{B,C} : \mathcal{C}(C \times A, B) \xrightarrow{\cong} \mathcal{C}(C, B^A)$$

natural in B, C . In that case, $\text{ev}_{A,B} := \Lambda^{-1}(\text{id}_{B^A})$.

Exercise 75 Derive $_^A$ and A^{-1} of the above description from **ev** and λ of definition 74.

Exercise 76 Show that \mathcal{C} has exponentials iff, for every $A, B, C \in \text{Ob}(\mathcal{C})$, there is an object B^A and a bijection

$$\theta_C : \mathcal{C}(C \times A, B) \xrightarrow{\cong} \mathcal{C}(C, B^A)$$

natural in C .

Notation 77 The notation B^A for exponential objects is standard in the category theory literature. For our purposes, however, it will be more convenient to write $A \Rightarrow B$.

Exponentials bring us to another fundamental notion, this time for understanding functional types, models of λ -calculus, and the structure of proofs.

Definition 78 A category with a terminal object, products and exponentials is called a **Cartesian Closed Category (CCC)**. ▲

For example, **Set** is a CCC. Another class of examples are *Boolean algebras*, seen as categories:

- Products are given by conjunctions $A \wedge B$. We define exponentials as *implications*:

$$A \Rightarrow B := \neg A \vee B.$$

- Evaluation is just *Modus Ponens*,

$$(A \Rightarrow B) \wedge A \leq B$$

while couniversality is the *Deduction Theorem*,

$$C \wedge A \leq B \iff C \leq A \Rightarrow B.$$

1.5.5 Exercises

1. Suppose that $U : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint F_1 , and $V : \mathcal{D} \rightarrow \mathcal{E}$ has a left adjoint F_2 . Show that $V \circ U : \mathcal{C} \rightarrow \mathcal{E}$ has a left adjoint.
2. A *sup-lattice* is a poset P in which every subset $S \subseteq P$ has a supremum (least upper bound) $\bigvee S$. Let P, Q be sup-lattices, and $f : P \rightarrow Q$ be a monotone map.
 - a. Show that if f has a right adjoint then f preserves least upper bounds:

$$f(\bigvee S) = \bigvee \{f(x) \mid x \in S\}.$$

- b. Show that if f preserves least upper bounds then it has a right adjoint g , given by:

$$g(y) = \bigvee \{x \in P \mid f(x) \leq y\}.$$

- c. Dualise to get a necessary and sufficient condition for the existence of left adjoints.

3. Let $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors such that F is left adjoint to G , with natural bijection $\theta_{C,D} : \mathcal{C}(C, GD) \xrightarrow{\cong} \mathcal{D}(FC, D)$. Show that there is a natural transformation $\varepsilon : F \circ G \rightarrow \text{Id}_{\mathcal{D}}$, the **counit** of the adjunction. Describe this counit explicitly in the case where the right adjoint is the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$.
4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors, and assume F is left adjoint to G with natural bijection θ .
- Show that F preserves epimorphisms.
 - Show that F is faithful if and only if, for every object A of \mathcal{C} , $\eta_A : A \rightarrow GF(A)$ is monic.
 - Show that if, for each object A of \mathcal{C} , there is a morphism $s_A : GF(A) \rightarrow A$ such that $\eta_A \circ s_A = \text{id}_{GF(A)}$ then F is full.

1.6 The Curry-Howard Correspondence

We shall now study a beautiful three-way connection between logic, computation and categories:

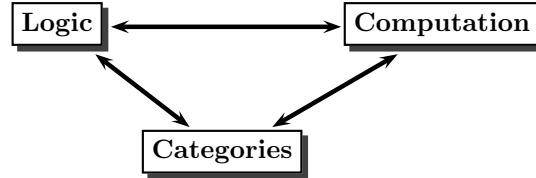


Table 1.2: The Curry-Howard Correspondence.

This connection has been known since the 1970's, and is widely used in Computer Science—it is also beginning to be used in Quantum Informatics! It is the upper link (Logic–Computation) that is usually attributed to Haskell B. Curry and William A. Howard, although the idea is related to the operational interpretation of intuitionistic logic given in various formulations by Brouwer, Heyting and Kolmogorov. The link to Categories is mainly due to the pioneering work of Joachim Lambek.

1.6.1 Logic

Suppose we ask ourselves the question: *What is Logic about?* There are two main kinds of answer: one focuses on *Truth*, and the other on *Proof*. We focus on the latter, that is, on:

What follows from what

Traditional introductions to logic focus on Hilbert-style proof systems, that is, on generating the set of *theorems* of a system from a set of *axioms* by applying rules of inference (e.g. Modus Ponens).

A key step in logic took place in the 1930's with the advent of *Gentzen-style systems*. Instead of focusing on theorems, we look more generally and symmetrically at *What follows from what*: in these systems the primary focus is on *proofs from assumptions*. We will examine two such kinds of systems: Natural Deduction systems and Gentzen sequent calculi.

Definition 79 Consider the fragment of propositional logic with logical connectives \wedge and \supset . The assertion that a formula A can be proved from assumptions A_1, \dots, A_n is expressed by a **sequent**:

$$A_1, \dots, A_n \vdash A$$

We use Γ, Δ to range over finite sets of formulas, and write Γ, A for $\Gamma \cup \{A\}$. Proofs are built using the proof rules of table 1.3; the resulting proof system is called *the Natural Deduction system for \wedge, \supset* . ▲

Identity	Conjunction	Implication
$\frac{}{\Gamma, A \vdash A} \text{Id}$	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge \text{intro}$ $\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge \text{elim}_1$ $\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge \text{elim}_2$	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset \text{intro}$ $\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset \text{elim}$

Table 1.3: Natural Deduction System for \wedge, \supset .

For example, we have the following proof of \supset -transitivity.

$$\begin{array}{c}
 \frac{}{A \supset B, B \supset C, A \vdash B \supset C} \text{Id} \quad \frac{A \supset B, B \supset C, A \vdash A \supset B}{A \supset B, B \supset C, A \vdash B} \text{Id} \quad \frac{A \supset B, B \supset C, A \vdash A}{A \supset B, B \supset C, A \vdash B} \text{Id} \\
 \frac{}{A \supset B, B \supset C, A \vdash B \supset C} \text{Id} \quad \frac{A \supset B, B \supset C, A \vdash B \supset C \quad A \supset B, B \supset C, A \vdash B}{A \supset B, B \supset C, A \vdash C} \supset \text{E} \\
 \frac{A \supset B, B \supset C, A \vdash C}{A \supset B, B \supset C \vdash A \supset C} \supset \text{I}
 \end{array}$$

An important feature of Natural Deduction is the systematic pattern it exhibits in the structure of the inference rules. For each connective \square there are *introduction rules*, which show how formulas $A\square B$ can be derived, and *elimination rules*, which show how such formulas can be used to derive other formulas.

Admissibility

We say that a proof rule

$$\frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_n \vdash A_n}{\Delta \vdash B}$$

is **admissible** in Natural Deduction if, whenever there are proofs of $\Gamma_i \vdash A_i$ then there is also a proof of $\Delta \vdash B$. For example, the following *Cut rule* is admissible.

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \text{ Cut}$$

Exercise 80 Show that the following rules are admissible in Natural Deduction.

1. The Weakening rule:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

2. The Cut rule.

Our focus will be on **Structural Proof Theory**, that is studying the “space of formal proofs” as a mathematical structure in its own right, rather than focussing only on

$$\text{Provability} \longleftrightarrow \text{Truth}$$

(*i.e.* the usual notions of “soundness and completeness”). One motivation for this approach comes from trying to understand and use the *computational content* of proofs, epitomised in the “Curry-Howard correspondence”.

1.6.2 Computation

Our starting point in computation is the pure calculus of functions called the *λ -calculus*.

Definition 81 Assume a countably infinite set of **variables**, ranged over by x, y, z and variants. **λ -calculus terms**, ranged over by t, u, v etc, are constructed from the following inductive definition.

- Every variable x is a term.
- If t and u are terms, then $t u$ is a term (**application**).
- If x is a variable and t is a term, then $\lambda x. t$ is a term (**λ -abstraction**). \blacktriangle

The above definition can be given in the following compact form, which will be followed in similar definitions in the sequel.

$$\begin{aligned} \text{VA} &\ni x, y, z, \dots \\ \text{TE} &\ni t, u, v ::= x \mid t u \mid \lambda x. t \end{aligned}$$

The computational content of the calculus is exhibited in the following examples. Note that the first example is not part of our formal syntax: it presupposes some *encoding* of numerals and successors.

$\lambda x. x + 1$	successor function
$\lambda x. x$	identity function
$\lambda f. \lambda x. f x$	application
$\lambda f. \lambda x. f(f x)$	double application
$\lambda f. \lambda g. \lambda x. g(f(x))$	composition and application

Table 1.4: Examples of λ -terms.

What we also note above is the use of parentheses in order to disambiguate the structure of terms (*i.e.* the precedence of term constructors). To avoid notational clutter we also use the following conventions.

- Applications associate to the left. For example, $f x y$ stands for $(f x) y$.
- The scope of an abstractions goes as far to the right as possible. For example,

$$\lambda f. (\lambda x. f(x x)) \lambda x. f(x x) \text{ stands for } \lambda f. ((\lambda x. (f(x x))) (\lambda x. (f(x x)))) .$$

The *free variables* of a term are those that are not bound by any λ ; they can be seen as the *assumptions* of the term.

Definition 82 The set of **free variables** of a term t , $\text{fv}(t)$, is given by:

$$\begin{aligned} \text{fv}(x) &:= \{x\}, \\ \text{fv}(t u) &:= \text{fv}(t) \cup \text{fv}(u), \\ \text{fv}(\lambda x. t) &:= \text{fv}(t) \setminus \{x\}. \end{aligned}$$

\blacktriangle

The notation $\lambda x. t$ is meant to serve the purpose of expressing formally

the function that returns t on input x .

Thus, λ is a *binder*, that is, it binds the variable x in the ‘function’ $\lambda x. t$, in the same way that e.g. \int binds x in $\int f(x) dx$. This means that there

should be no difference between $\lambda x.t$ and $\lambda x'.t'$, where t' is obtained from t by swapping x with some *fresh* variable x' (i.e. with some x' not appearing free in t). For example, the terms

$$\lambda x.x \text{ and } \lambda x'.x'$$

should be ‘equal’, as they both stand for the identity function. We formalise this by stipulating that

Terms are identified up to α -equivalence

where we say that two terms are α -equivalent iff they differ solely in the choice of variables appearing in *binding positions*. This is formally defined in two steps, as follows.

Definition 83 We define *variable-swapping* on terms recursively as follows.

$$\begin{aligned} (y \ x) \bullet z &:= \begin{cases} y & \text{if } z = x \\ x & \text{if } z = y \\ z & \text{otherwise} \end{cases} \\ (y \ x) \bullet t \ u &:= ((y \ x) \bullet t)((y \ x) \bullet u) \\ (y \ x) \bullet \lambda z.t &:= \lambda((y \ x) \bullet z).((y \ x) \bullet t) \end{aligned}$$

Then, α -**equivalence**, $=_\alpha$, is the relation on terms defined inductively by:⁴

- $x =_\alpha x$,
- $t \ u =_\alpha t' \ u'$ if $t =_\alpha t'$ and $u =_\alpha u'$,
- $\lambda x.t =_\alpha \lambda x'.t'$ if, for all y not appearing in $t \ t'$, $(y \ x) \bullet t =_\alpha (y \ x') \bullet t'$. \blacktriangle

Equating terms modulo α -equivalence means that we work with $\text{TE}/=_\alpha$ instead of TE . Henceforth, we will refer to elements of $\text{TE}/=_\alpha$ as *terms*, and to elements of TE as *raw terms*. Note that α -equivalence is meaningful only on raw terms.

Exercise 84 Prove the following α -equivalences.

$$\lambda x.x =_\alpha \lambda y.y, \quad \lambda x.\lambda y.xy =_\alpha \lambda y.\lambda x.yx, \quad x(\lambda x.x) =_\alpha x(\lambda y.y).$$

Exercise 85 Show that, for all raw terms t, t' and variables x, x' , if $t =_\alpha t'$ then $\text{fv}(t) = \text{fv}(t')$ and $(x \ x') \bullet t =_\alpha (x \ x') \bullet t'$.

⁴ The last clause can be replaced by any of the following:

- ... if, for some y not appearing in $t \ t'$, $(y \ x) \bullet t =_\alpha (y \ x') \bullet t'$.
- ... if, for all y not appearing free in $t \ t'$, $(y \ x) \bullet t =_\alpha (y \ x') \bullet t'$.
- ... if, for some y not appearing free in $t \ t'$, $(y \ x) \bullet t =_\alpha (y \ x') \bullet t'$.

Moreover, show that, for any $x, x' \notin \text{fv}(t)$, $t =_\alpha (x \ x') \bullet t$. Hence infer that, for any $x' \notin \text{fv}(t)$, $\lambda x. t =_\alpha \lambda x'. (x \ x') \bullet t$.

From the above exercise we obtain that fv and variable-swapping extend to terms (*i.e.* to $\text{TE}/=_\alpha$) in a straightforward manner. Moreover, we have that, for any term t and any $x' \notin \text{fv}(t)$,

$$\lambda x. t = \lambda x'. (x \ x') \bullet t.$$

Since λ -abstractions stand for functions, an application of a λ -abstraction on another term should result to a *substitution* of the latter inside the body of the abstraction.

Definition 86 Define the *substitution* of a term t for a variable x inside a term inductively by:

$$\begin{aligned} y[t/x] &:= \begin{cases} t & \text{if } y = x \\ y & \text{if } y \neq x \end{cases} \\ (uv)[t/x] &:= (u[t/x])(v[t/x]) \\ (\lambda z. u)[t/x] &:= \lambda z. (u[t/x]) \quad (*) \end{aligned}$$

where $(*)$ indicates the condition that $z \notin \text{fv}(xt)$. ▲

Note that, due to identification of α -equivalent (raw) terms, it is always possible to rename bound variables so that condition $(*)$ be satisfied: for example,

$$(\lambda z. zx)[z/x] = (\lambda y. yx)[z/x] = \lambda y. yz$$

Exercise 87 Show that, for all λ -terms u, t, t' and variables x, x' such that $x' \notin \text{fv}(u) \setminus \{x\}$,

$$u[t/x][t'/x'] = u[(t[t'/x'])/x].$$

We proceed to the definition of β -reduction and β -conversion. These are relations defined on pairs of terms and express the computational content of the calculus.

Definition 88 We take β -*reduction*, \longrightarrow_β , to be the relation defined by:

$$(\lambda x. t) u \longrightarrow_\beta t[u/x].$$

This extends to arbitrary terms as follows. If $t \longrightarrow_\beta t'$ then:

$$t u \longrightarrow_\beta t' u, \quad u t \longrightarrow_\beta u t', \quad \lambda x. t \longrightarrow_\beta \lambda x. t'.$$

We take β -*conversion*, $=_\beta$, to be the symmetric reflexive transitive closure of β -reduction, that is, the equivalence relation induced by:

$$(\lambda x.t) u =_{\beta} t[u/x].$$

▲

With β -reduction we obtain a notion of “computational dynamics”. For example:

$$\begin{aligned} (\lambda f.f(f y))(\lambda x.x + 1) &\longrightarrow_{\beta} (\lambda x.x + 1)((\lambda x.x + 1) y) \\ &\longrightarrow_{\beta} ((\lambda x.x + 1) y) + 1 \longrightarrow_{\beta} (y + 1) + 1 \\ (\lambda f.f(f y))(\lambda x.x + 1) &\longrightarrow_{\beta} (\lambda x.x + 1)((\lambda x.x + 1) y) \\ &\longrightarrow_{\beta} (\lambda x.x + 1)(y + 1) \longrightarrow_{\beta} (y + 1) + 1 \end{aligned}$$

Note that in the sequel we will usually write β -reduction simply by “ \longrightarrow ”.

1.6.3 Simply-Typed λ -calculus

The ‘pure’ λ -calculus we have discussed so far is *very* unconstrained. For example, it allows *self-application*, *i.e.* terms like xx are perfectly legal. On the one hand, this means that the calculus is very expressive: for example, we can encode *recursion* by setting

$$\mathbf{Y} := \lambda f.(\lambda x.f(xx)) \lambda x.f(xx).$$

We have:

$$\mathbf{Y}t \longrightarrow (\lambda x.t(xx)) \lambda x.t(xx) \longrightarrow t((\lambda x.t(xx)) \lambda x.t(xx)) \longleftarrow t(\mathbf{Y}t)$$

However, self-application leads also to *divergences*. The most characteristic example is the following. Setting $\Omega := (\lambda x.xx) \lambda x.xx$, we have:

$$\Omega \longrightarrow \Omega \longrightarrow \Omega \longrightarrow \dots$$

Historically, Curry extracted \mathbf{Y} from an analysis of Russell’s Paradox, so it should come as no surprise that it too leads to divergences: setting t' to be $\lambda x.t(xx)$,

$$\mathbf{Y}t \longrightarrow t't' \longrightarrow t(t't') \longrightarrow t(t(t't')) \longrightarrow \dots$$

The solution is to introduce *types*. The original idea, due to Church following Russell, was that:

Types are there to stop you doing bad things

However, it has turned out that types constitute one of the most fruitful *positive* ideas in Computer Science, and provide one of the key disciplines of programming.

Definition 89 Let us assume a set of *base types*, ranged over by b . The *simply-typed λ -calculus* is defined as follows.

Type	$\text{TY} \ni T, U ::= b \mid T \rightarrow U \mid T \times U$
Term	$\text{TE} \ni t, u ::= x \mid tu \mid \lambda x. t \mid \langle t, u \rangle \mid \pi_1 u \mid \pi_2 u$
Typing context	$\Gamma ::= \emptyset \mid x : T, \Gamma \quad (x \text{ does not appear in } \Gamma)$

A *typing judgement* is a triple of the form

$$\Gamma \vdash t : T,$$

which is to be understood as the assertion that term t has the type T under the assumptions that x_1 has type T_1, \dots, x_k has type T_k , if $\Gamma = x_1 : T_1, \dots, x_k : T_k$. A *typed term* is a term t accompanied with a type T and a context Γ , such that the judgement $\Gamma \vdash t : T$ is derivable by use of the *typing rules* of table 1.5. \blacktriangle

Variable	Product	Function
$\frac{}{\Gamma, x : T \vdash x : T}$	$\frac{\Gamma \vdash t : T \quad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}$	$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \rightarrow T}$
	$\frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T}$	$\frac{\Gamma \vdash t : U \rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$
	$\frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$	

Table 1.5: Simply-Typed λ -calculus for \times, \rightarrow .

Note that contexts are sets, and so $x : T, \Gamma$ stands for $\{x : T\} \cup \Gamma$ with x not appearing in Γ . As before, terms are identified up to α -equivalence.

From the definition of types we see that the simply-typed λ -calculus is a calculus of functions and products. For example:

$b \rightarrow b \rightarrow b$ first-order function type

$(b \rightarrow b) \rightarrow b$ second-order function type

Exercise 90 Can you type the following terms?

$$\lambda x. xx, \quad \lambda f. (\lambda x. f(xx))(\lambda x. f(xx)).$$

Exercise 91 (Weakening & Cut) Show that Weakening and Cut are admissible in the typing system of the simply-typed λ -calculus:

$$\frac{\Gamma \vdash t : T}{\Gamma, x : U \vdash t : T} \text{Weak} \qquad \frac{\Gamma \vdash t : T \quad \Gamma, x : T \vdash u : U}{\Gamma \vdash u[t/x] : U} \text{Cut}$$

We proceed to the rules for reduction and conversion. These are given as in the untyped case, with the addition of η -rules, which are essentially extensionality principles.

Definition 92 We define β -reduction, \longrightarrow_β , by the following rules, and let β -conversion, $=_\beta$, be its symmetric reflexive transitive closure.

$$\begin{aligned} (\lambda x. t)u &\longrightarrow_\beta t[u/x] \\ \pi_1 \langle t, u \rangle &\longrightarrow_\beta t \\ \pi_2 \langle t, u \rangle &\longrightarrow_\beta u \end{aligned}$$

Moreover, η -conversion, $=_\eta$, is the symmetric reflexive transitive relation obtained by the following rules,

$$\begin{aligned} t &=_\eta \lambda x. tx && x \notin \text{fv}(t), \text{ at function types} \\ v &=_\eta \langle \pi_1 v, \pi_2 v \rangle && \text{ at product types} \end{aligned}$$

and λ -conversion, $=_\lambda$, is the transitive closure of $=_\beta \cup =_\eta$. ▲

Implicit in the above definition is the fact that η -rules relate typed terms. For example, $t =_\eta \lambda x. tx$ has as side condition that t be of function type, *i.e.* that t be a typed term $\Gamma \vdash t : T \rightarrow U$. Now, following our intuitive interpretation of arrows as functions, we can read this η -rule as:

t is the function returning $t(x)$ to every input x

Note that the above statement is in fact the couniversal property of currying in **Set**; we will see more on this in the next sections!

Exercise 93 (Subject Reduction) Show that, for any typed term $\Gamma \vdash t : T$, if $t \longrightarrow_\beta t'$ then $\Gamma \vdash t' : T$ is derivable.

Strong Normalisation

Term reduction results in a *normal form*: an *explicit but much longer expression* in which no more reductions are applicable. Formally, a λ -term is called a **redex** if it is in one of forms of the left-hand-side of the β -reduction rules, and therefore β -reduction can be applied to it. A term is in **normal form** if it contains no redexes. In the light of the correspondence presented in the next paragraph, a term in normal form corresponds to a proof in which all lemmas have been eliminated.

Fact 94 (SN) *For every term t , there is no infinite sequence of β -reductions:*

$$t \longrightarrow t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow \cdots$$

The above result states that *every* reduction sequence leads eventually to a term in normal form. Note, though, that reduction to normal form has enormous (*non-elementary*) complexity.

The correspondence between Logic and Computation

Comparing the following two systems,

Natural Deduction System for \wedge, \supset	vs	Simply-Typed λ -calculus for \times, \rightarrow
--	----	--

we notice that if we equate

$$\begin{aligned}\wedge &\equiv \times \\ \supset &\equiv \rightarrow\end{aligned}$$

then they are the same! This is the Logic–Computation part of the Curry–Howard correspondence (sometimes: “Curry–Howard isomorphism”). It works on three levels:

Natural Deduction System	Simply-Typed λ -calculus
Formulas	Types
Proofs	Terms
Proof transformations	Term reductions

Table 1.6: Correspondence between Logic and Computation.

The view of proofs as containing computational content can also be detected in the *Brouwer–Heyting–Kolmogorov interpretation* of intuitionistic logic:

- A proof of an implication $A \supset B$ is a procedure which transforms any proof of A into a proof of B .
- A proof of $A \wedge B$ is a pair consisting of a proof of A and a proof of B .

These readings motivate identifying $A \wedge B$ with $A \times B$, and $A \supset B$ with $A \rightarrow B$. Moreover, these ideas have strong connections to computing. The λ -calculus is a ‘pure’ version of functional programming languages such as Haskell and SML. So we get a reading of:

<i>Proofs as Programs</i>

1.6.4 Categories

We now have our link between Logic and Computation. We now proceed to complete the triangle of the Curry-Howard correspondence by showing the connection to Categories.

We establish the link from Logic (and Computation) to Categories. Let \mathcal{C} be a cartesian closed category. We shall interpret formulas (or types) as objects of \mathcal{C} . A morphism $f : A \rightarrow B$ will then correspond to a proof of B from assumption A , *i.e.* a proof of $A \vdash B$ (a typed term $x : A \vdash t : B$). Note that the bare structure of a category only supports proofs from a single assumption. Since \mathcal{C} has finite products, a proof of

$$A_1, \dots, A_k \vdash A$$

will correspond to a morphism

$$f : A_1 \times \dots \times A_k \longrightarrow A.$$

The correspondence is depicted in the next table.

Axiom	$\overline{\Gamma, A \vdash A}^{\text{Id}}$	$\overline{\pi_2 : \Gamma \times A \longrightarrow A}$
Conjunction	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I$	$\frac{f : \Gamma \longrightarrow A \quad g : \Gamma \longrightarrow B}{\langle f, g \rangle : \Gamma \longrightarrow A \times B}$
	$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_1$	$\frac{f : \Gamma \longrightarrow A \times B}{\pi_1 \circ f : \Gamma \longrightarrow A}$
	$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_2$	$\frac{f : \Gamma \longrightarrow A \times B}{\pi_2 \circ f : \Gamma \longrightarrow B}$
Implication	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset I$	$\frac{f : \Gamma \times A \longrightarrow B}{A(f) : \Gamma \longrightarrow (A \Rightarrow B)}$
	$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset E$	$\frac{f : \Gamma \longrightarrow (A \Rightarrow B) \quad g : \Gamma \longrightarrow A}{\text{ev}_{A,B} \circ \langle f, g \rangle : \Gamma \longrightarrow B}$

Table 1.7: Correspondence between Logic and Categories.

Moreover, the rules for β - and η -conversion are all then *derivable* from the equations of cartesian closed categories. So cartesian closed categories are *models* of \wedge, \supset -logic at the level of proofs and proof-transformations, and of simply typed λ -calculus at the level of terms and term-conversions. The connection to computation is examined in more detail below.

Remark 95 In our translation of Logic sequents there is an implicit ordering of assumptions: a *set* of assumptions is mapped to an assumption *product*,

$$\{A_1, \dots, A_n\} \mapsto A_1 \times \dots \times A_n.$$

In practice, since for any permutation A'_1, \dots, A'_n of A_1, \dots, A_n we have

$$A_1 \times \dots \times A_n \cong A'_1 \times \dots \times A'_n,$$

such an ordering is harmless.

1.6.5 Categorical Semantics of Simply-Typed λ -calculus

We translate the simply-typed λ -calculus into a cartesian closed category \mathcal{C} , so that to each typed term $x_1 : T_1, \dots, x_k : T_k \vdash t : T$ corresponds an arrow

$$\llbracket t \rrbracket : \llbracket T_1 \rrbracket \times \dots \times \llbracket T_k \rrbracket \longrightarrow \llbracket T \rrbracket.$$

The translation is given by the function $\llbracket _ \rrbracket$ defined below (“semantic brackets”).

Definition 96 (Semantic translation) Let \mathcal{C} be a CCC and suppose we are given an assignment of an object \tilde{b} to each base type b . Then, the translation is defined recursively on types by:

$$\llbracket b \rrbracket := \tilde{b}, \quad \llbracket T \times U \rrbracket := \llbracket T \rrbracket \times \llbracket U \rrbracket, \quad \llbracket T \rightarrow U \rrbracket := \llbracket T \rrbracket \Rightarrow \llbracket U \rrbracket,$$

and on typed terms by:

$$\begin{array}{c} \overline{\llbracket \Gamma, x : T \vdash x : T \rrbracket := \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket T \rrbracket \longrightarrow \llbracket T \rrbracket} \\ \overline{\llbracket \Gamma \vdash t : T \times U \rrbracket = f : \llbracket \Gamma \rrbracket \longrightarrow \llbracket T \rrbracket \times \llbracket U \rrbracket} \\ \llbracket \Gamma \vdash \pi_1 t : T \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket T \rrbracket \times \llbracket U \rrbracket \xrightarrow{\pi_1} \llbracket T \rrbracket \\ \overline{\llbracket \Gamma \vdash t : T \rrbracket = f : \llbracket \Gamma \rrbracket \longrightarrow \llbracket T \rrbracket \quad \llbracket \Gamma \vdash u : U \rrbracket = g : \llbracket \Gamma \rrbracket \longrightarrow \llbracket U \rrbracket} \\ \llbracket \Gamma \vdash \langle t, u \rangle : T \times U \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\langle f, g \rangle} \llbracket T \rrbracket \times \llbracket U \rrbracket \\ \overline{\llbracket \Gamma, x : T \vdash t : U \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket T \rrbracket \longrightarrow \llbracket U \rrbracket} \\ \llbracket \Gamma \vdash \lambda x. t : T \rightarrow U \rrbracket := \Lambda(f) : \llbracket \Gamma \rrbracket \longrightarrow (\llbracket T \rrbracket \Rightarrow \llbracket U \rrbracket) \\ \overline{\llbracket \Gamma \vdash t : T \rightarrow U \rrbracket = f \quad \llbracket \Gamma \vdash u : T \rrbracket = g} \\ \llbracket \Gamma \vdash t u : U \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\langle f, g \rangle} (\llbracket T \rrbracket \Rightarrow \llbracket U \rrbracket) \times \llbracket T \rrbracket \xrightarrow{\text{ev}} \llbracket U \rrbracket \end{array}$$



Our aim now is to verify that λ -conversion (induced by β - and η -rules) is *preserved* by the translation, *i.e.* that, for any t, u ,

$$t =_{\lambda} u \implies \llbracket t \rrbracket = \llbracket u \rrbracket.$$

This would mean that our categorical semantics is *sound*.

Let us recall some structures from CCC's. Given $f_1 : D_1 \rightarrow E_1$, $f_2 : D_2 \rightarrow E_2$, we defined

$$f_1 \times f_2 = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle : D_1 \times D_2 \longrightarrow E_1 \times E_2,$$

and we showed that $(f_1 \times f_2) \circ \langle h_1, h_2 \rangle = \langle f_1 \circ h_1, f_2 \circ h_2 \rangle$. Moreover, exponentials are given by the following natural bijection.

$$\frac{f : D \times E \longrightarrow F}{\Lambda(f) : D \longrightarrow (E \Rightarrow F)}$$

Equivalently, recall the basic equation:

$$\text{ev} \circ (\Lambda(f) \times \text{id}_E) = f,$$

where $\Lambda(f)$ is the unique arrow $D \rightarrow (E \Rightarrow F)$ satisfying this equation, with uniqueness being specified by:

$$\forall h : D \longrightarrow (E \Rightarrow F). \Lambda(\text{ev} \circ (h \times \text{id}_E)) = h.$$

Naturality of Λ is then proven as follows.

Proposition 97 *For any $f : A \times B \rightarrow C$ and $g : A' \rightarrow A$,*

$$\Lambda(f) \circ g = \Lambda(f \circ (g \times \text{id}_B)).$$

Proof:

$$\begin{aligned} \Lambda(f) \circ g &= \Lambda(\text{ev} \circ ((\Lambda(f) \circ g) \times \text{id}_B)) \\ &= \Lambda(\text{ev} \circ (\Lambda(f) \times \text{id}_B) \circ (g \times \text{id}_B))) = \Lambda(f \circ (g \times \text{id}_B)). \end{aligned} \quad \blacksquare$$

Substitution Lemma

We consider a *simultaneous substitution* for all the free variables in a term.

Definition 98 Let $\Gamma = x_1 : T_1, \dots, x_k : T_k$. Given typed terms

$$\Gamma \vdash t : T \quad \text{and} \quad \Gamma \vdash t_i : T_i, \quad 1 \leq i \leq k,$$

we define $t[\mathbf{t}/\mathbf{x}] \equiv t[t_1/x_1, \dots, t_k/x_k]$ recursively by:

$$\begin{aligned} x_i[\mathbf{t}/\mathbf{x}] &:= t_i \\ (\pi_i t)[\mathbf{t}/\mathbf{x}] &:= \pi_i(t[\mathbf{t}/\mathbf{x}]) \\ \langle t, u \rangle[\mathbf{t}/\mathbf{x}] &:= \langle t[\mathbf{t}/\mathbf{x}], u[\mathbf{t}/\mathbf{x}] \rangle \\ (tu)[\mathbf{t}/\mathbf{x}] &:= (t[\mathbf{t}/\mathbf{x}](u[\mathbf{t}/\mathbf{x}])) \\ (\lambda x. t)[\mathbf{t}/\mathbf{x}] &:= \lambda x. t[\mathbf{t}, x/\mathbf{x}, x]. \end{aligned}$$

▲

Note that, in contrast to ordinary substitution, simultaneous substitution can be defined directly on raw terms, that is, prior to equating them modulo α -equivalence. Moreover, we can show that:

$$t[t_1/x_1, \dots, t_k/x_k] = t[t_1/x_1] \cdots [t_k/x_k].$$

We can now show the following *Substitution Lemma*.

Proposition 99 For t, t_1, \dots, t_k as in the previous definition,

$$\llbracket t[t_1/x_1, \dots, t_k/x_k] \rrbracket = \llbracket t \rrbracket \circ \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_k \rrbracket \rangle.$$

Proof: By induction on the structure of t .

(1) If $t = x_i$:

$$\llbracket x_i[\mathbf{t}/\mathbf{x}] \rrbracket = \llbracket t_i \rrbracket = \pi_i \circ \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_k \rrbracket \rangle = \llbracket x_i \rrbracket \circ \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_k \rrbracket \rangle.$$

(2) If $t = uv$ then, abbreviating $\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_k \rrbracket \rangle$ to $\langle \llbracket \mathbf{t} \rrbracket \rangle$ we have:

$$\begin{aligned} \llbracket uv[\mathbf{t}/\mathbf{x}] \rrbracket &= \llbracket (u[\mathbf{t}/\mathbf{x}](v[\mathbf{t}/\mathbf{x}])) \rrbracket && \text{Defn of substitution} \\ &= \mathbf{ev} \circ \langle \llbracket u[\mathbf{t}/\mathbf{x}] \rrbracket, \llbracket v[\mathbf{t}/\mathbf{x}] \rrbracket \rangle && \text{Defn of semantic function} \\ &= \mathbf{ev} \circ \langle \llbracket u \rrbracket \circ \langle \llbracket \mathbf{t} \rrbracket \rangle, \llbracket v \rrbracket \circ \langle \llbracket \mathbf{t} \rrbracket \rangle \rangle && \text{Induction hyp.} \\ &= \mathbf{ev} \circ \langle \llbracket u \rrbracket, \llbracket v \rrbracket \rangle \circ \langle \llbracket \mathbf{t} \rrbracket \rangle && \text{Property of products} \\ &= \llbracket uv \rrbracket \circ \langle \llbracket \mathbf{t} \rrbracket \rangle && \text{Defn of semantic function} \end{aligned}$$

(3) If $t = \lambda x. u$:

$$\begin{aligned} \llbracket \lambda x. u[\mathbf{t}/\mathbf{x}] \rrbracket &= \llbracket \lambda x. (u[\mathbf{t}, x/\mathbf{x}, x]) \rrbracket && \text{Defn. of substitution} \\ &= \Lambda(\llbracket u[\mathbf{t}, x/\mathbf{x}, x] \rrbracket) && \text{Defn. of semantic function} \\ &= \Lambda(\llbracket u \rrbracket \circ (\langle \llbracket \mathbf{t} \rrbracket \rangle \times \text{id})) && \text{Induction hyp.} \\ &= \Lambda(\llbracket u \rrbracket) \circ \langle \llbracket \mathbf{t} \rrbracket \rangle && \text{Prop. 97} \\ &= \llbracket \lambda x. u \rrbracket \circ \langle \llbracket \mathbf{t} \rrbracket \rangle && \text{Defn. of semantic function} \end{aligned}$$

(4,5) The cases of projections and pairs are left as exercise. ■

Exercise 100 Complete the proof of the above proposition.

Validating the conversion rules

We can now show that the conversion rules of the λ -calculus are preserved by the translation, and hence the interpretation is sound. Observe the correspondence between η -rules and uniqueness (couniversality) principles.

- For β -conversion: $[(\lambda x. t)u = t[u/x], \pi_1 \langle t, u \rangle = t, \pi_2 \langle t, u \rangle = u]$

$$\begin{aligned}
 \llbracket (\lambda x. t)u \rrbracket &= \text{ev} \circ \langle \Lambda(\llbracket t \rrbracket), \llbracket u \rrbracket \rangle && \text{Defn. of semantics} \\
 &= \text{ev} \circ (\Lambda(\llbracket t \rrbracket) \times \text{id}) \circ \langle \text{id}_{\llbracket t \rrbracket}, \llbracket u \rrbracket \rangle && \text{Property of } \times \\
 &= \llbracket t \rrbracket \circ \langle \text{id}_{\llbracket t \rrbracket}, \llbracket u \rrbracket \rangle && \text{Defn. of } \Lambda \\
 &= \llbracket t[\mathbf{x}, u/\mathbf{x}, x] \rrbracket && \text{Substitution lemma.}
 \end{aligned}$$

$$\llbracket \pi_1 \langle t, u \rangle \rrbracket = \pi_1 \circ \llbracket \langle t, u \rangle \rrbracket = \pi_1 \circ \langle \llbracket t \rrbracket, \llbracket u \rrbracket \rangle = \llbracket t \rrbracket.$$

- For η -conversion: $[t = \lambda x. tx, \langle \pi_1 t, \pi_2 t \rangle = t]$

$$\begin{aligned}
 \llbracket \lambda x. tx \rrbracket &= \Lambda(\text{ev} \circ (\llbracket t \rrbracket \times \text{id})) = \llbracket t \rrbracket && \text{Uniqueness equation } (\Rightarrow) \\
 \llbracket \langle \pi_1 t, \pi_2 t \rangle \rrbracket &= \langle \pi_1 \circ \llbracket t \rrbracket, \pi_2 \circ \llbracket t \rrbracket \rangle = \llbracket t \rrbracket && \text{Uniqueness equation } (\times)
 \end{aligned}$$

1.6.6 Completeness?

It is the case that, in a general CCC \mathcal{C} , there may be equalities which are *not* reflected by the semantic translation, *i.e.*

$$\llbracket t \rrbracket = \llbracket u \rrbracket \quad \text{yet} \quad t \neq_\lambda u.$$

In the rest of this section, we show how to *construct* a CCC \mathcal{C}_λ in which equalities between arrows correspond precisely to λ -conversions between terms. We call \mathcal{C}_λ a *term model*, due to its dependence on the syntax.

Definition 101 We define a family of relations on variable-term pairs by setting $(x, t) \sim_{T,U} (y, u)$ if $x : T \vdash t : U$ and $y : T \vdash u : U$ are derivable and

$$t =_\lambda u[x/y].$$

These are equivalence relations, so we set:

$$[(x, t)]_{T,U} := \{ (y, u) \mid (x, t) \sim_{T,U} (y, u) \}.$$

Similarly, $(\cdot, t) \sim_{\cdot, U} (\cdot, u)$ if $\vdash t : U$ and $\vdash u : U$ are derivable and $t =_\lambda u$. Moreover,

$$[(\cdot, t)]_{\cdot, U} := \{ (\cdot, u) \mid (\cdot, t) \sim_{\cdot, U} (\cdot, u) \}.$$



We denote $[(x, t)]_{T, U}$ simply by $[x, t]$, and $[(\bullet, t)]_{\bullet, U}$ simply by $[\bullet, t]$ (these are *not* to be confused with copairings!). We proceed with \mathcal{C}_λ .

Definition 102 The category \mathcal{C}_λ is defined as follows. We take as set of objects the set of λ -types augmented with a terminal object $\mathbf{1}$:

$$Ob(\mathcal{C}_\lambda) := \{\mathbf{1}\} \cup \{ \tilde{T} \mid T \text{ a } \lambda\text{-type} \}$$

The homsets of \mathcal{C}_λ contain equivalence relations on typed terms (definition 101), or terminal arrows τ :

$$\mathcal{C}_\lambda(\tilde{T}, \tilde{U}) := \{ [x, t] \mid x : T \vdash t : U \text{ is derivable} \}$$

$$\mathcal{C}_\lambda(\mathbf{1}, \tilde{U}) := \{ [\bullet, t] \mid \vdash t : U \text{ is derivable} \}$$

$$\mathcal{C}_\lambda(A, \mathbf{1}) := \{ \tau_A \}$$

The identities are:

$$\text{id}_{\tilde{T}} := [x, x], \quad \text{id}_{\mathbf{1}} := \tau_{\mathbf{1}},$$

and arrow composition is defined by:

$$[x, t] \circ [y, u] := [y, t[u/x]]$$

$$[x, t] \circ [\bullet, u] := [\bullet, t[u/x]]$$

$$[\bullet, t] \circ \tau_A := \begin{cases} [y, t] & \text{if } A = \tilde{U} \\ [\bullet, t] & \text{if } A = \mathbf{1} \end{cases}$$

$$\tau_B \circ h := \tau_A \quad (h \in \mathcal{C}_\lambda(A, B))$$



Note that, for each variable x' , any arrow $[x, t] : \tilde{T} \rightarrow \tilde{U}$ can be written in the form $[x', t']$, since $t = (t[x'/x])[x/x']$ and therefore $[x, t] = [x', t[x'/x]]$.

Proposition 103 \mathcal{C}_λ is a category.

Proof: It is not difficult to see that id 's are identities. For associativity, we show the most interesting case (and leave the rest as an exercise):

$$[x, t] \circ ([y, u] \circ [z, v]) = [x, t] \circ [z, u[v/y]] = [z, t[(u[v/y])/x]],$$

$$([x, t] \circ [y, u]) \circ [z, v] = [y, t[u/x]] \circ [z, v] = [z, t[u/x][v/y]].$$

By exercise 87, the above are equal. ■

Proposition 104 \mathcal{C}_λ has finite products.

Proof: Clearly, $\mathbf{1}$ is terminal with canonical arrows $\tau_A : A \rightarrow \mathbf{1}$. For (binary) products, $\mathbf{1} \times A = A \times \mathbf{1} = A$. Otherwise, define $\tilde{T} \xleftarrow{\pi_1} \tilde{T} \times \tilde{U} \xrightarrow{\pi_2} \tilde{U}$ by:

$$\begin{aligned}\tilde{T} \times \tilde{U} &:= \widetilde{T \times U} \\ \pi_i &:= [x, \pi_i x] \quad i = 1, 2.\end{aligned}$$

Given $\tilde{T} \xrightarrow{[x,t]} \tilde{V} \xrightarrow{[x,u]} \tilde{U}$, take $\langle [x, t], [x, u] \rangle : \tilde{V} \rightarrow \tilde{T} \times \tilde{U} := [x, \langle t, u \rangle]$. Then:

$$\begin{aligned}\pi_1 \circ \langle [x, t], [x, u] \rangle &= [y, \pi_1 y] \circ [x, \langle t, u \rangle] && \text{Definitions} \\ &= [x, \pi_1 \langle t, u \rangle] && \text{Defn of composition} \\ &= [x, t] && \beta\text{-conversion}\end{aligned}$$

Uniqueness left as exercise. The case of $\tilde{T} \xrightarrow{[\cdot, t]} \mathbf{1} \xrightarrow{[\cdot, u]} \tilde{U}$ is similar. \blacksquare

Proposition 105 \mathcal{C}_λ has exponentials.

Proof: We have that $\mathbf{1} \Rightarrow A = A$ and $A \Rightarrow \mathbf{1} = \mathbf{1}$, with obvious evaluation arrows. Otherwise,

$$\begin{aligned}\tilde{U} \Rightarrow \tilde{T} &:= \widetilde{U \rightarrow T} \\ \text{ev}_{\tilde{U}, \tilde{T}} : (\tilde{U} \Rightarrow \tilde{T}) \times \tilde{U} &\longrightarrow \tilde{T} := [x, (\pi_1 x)(\pi_2 x)]\end{aligned}$$

Given $[x, t] : \tilde{V} \times \tilde{U} \rightarrow \tilde{T}$, take $\Lambda([x, t]) := [x_1, \lambda x_2. t[\langle x_1, x_2 \rangle / x]]$. Then,

$$\begin{aligned}\text{ev} \circ \Lambda([x, t]) \times \text{id} &= \text{ev} \circ \langle \Lambda([x, t]) \circ \pi_1, \text{id} \circ \pi_2 \rangle \\ &= \text{ev} \circ \langle [x_1, \lambda x_2. t[\langle x_1, x_2 \rangle / x]] \circ [y, \pi_1 y], [y, \pi_2 y] \rangle \\ &= \text{ev} \circ \langle [y, \lambda x_2. t[\langle \pi_1 y, x_2 \rangle / x]], [y, \pi_2 y] \rangle \\ &= [z, (\pi_1 z)(\pi_2 z)] \circ [y, \langle \lambda x_2. t[\langle \pi_1 y, x_2 \rangle / x], \pi_2 y \rangle]_u \\ &= [y, (\pi_1 u)(\pi_2 u)] \stackrel{\beta}{=} [y, (\lambda x_2. t[\langle \pi_1 y, x_2 \rangle / x])(\pi_2 y)] \\ &\stackrel{\beta}{=} [y, t[\langle \pi_1 y, \pi_2 y \rangle / x]] \stackrel{\eta}{=} [y, t[y/x]] = [x, t].\end{aligned}$$

Uniqueness left as exercise. The case of $[x, t] : \mathbf{1} \times \tilde{U} \rightarrow \tilde{T}$ is similar. \blacksquare

Exercise 106 Complete the proof of the previous propositions.

Hence, \mathcal{C}_λ is a CCC and a sound model of the simply-typed λ -calculus. Moreover, applying our translation from the λ -calculus to a CCC (definition 96) we can show that we have

$$\llbracket \Gamma \vdash t : T \rrbracket = [x, t[\pi_i x / x_i]_{i=1..n}]$$

where $\Gamma = \{x_1 : T_1, \dots, x_n : T_n\}$, $x \notin \Gamma$ and $x : \prod_{i=1}^n T_i$. Then,

$$t =_\lambda u \iff \llbracket \Gamma \vdash t : T \rrbracket = \llbracket \Gamma \vdash u : T \rrbracket.$$

This means that our term model is *complete*.

1.6.7 Exercises

1. Give Natural Deduction proofs of the following sequents.

- $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$
- $\vdash (A \supset (A \supset B)) \supset (A \supset B)$
- $\vdash (C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$
- $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

In each case, give the corresponding λ -term and the corresponding arrow in a CCC \mathcal{C} .

2. For each of the following λ -terms, find a type for it. Try to find the ‘most general’ type, built from ‘type variables’ α, β etc. For example, the most general type for the identity $\lambda x. x$ is $\alpha \rightarrow \alpha$. In each case, give the derivation of the type for this term (where you may assume that types can be built up from type variables as well as base types).

- $\lambda f. \lambda x. fx$
- $\lambda x. \lambda y. \lambda z. x(yz)$
- $\lambda x. \lambda y. \lambda z. xzy$
- $\lambda x. \lambda y. xyy$
- $\lambda x. \lambda y. x$
- $\lambda x. \lambda y. \lambda z. xz(yz)$

Reflect a little on the methods you used to do this exercise. Could they be made algorithmic?

1.7 Linearity

In the system of Natural Deduction, implicit in our treatment of assumptions in sequents

$$A_1, \dots, A_n \vdash A$$

is that we can use them as many times as we want (including not at all). In this section we will explore the field that is opened once we apply restrictions to this approach, and thus render our treatment of assumptions more *linear* (or *resource sensitive*).

1.7.1 Gentzen Sequent Calculus

In order to make the manipulation of assumptions more visible, we now represent the assumptions as a *list* (possibly with repetitions) rather than a set, and use explicit structural rules to control copying, deletion and interchange of assumptions.

Definition 107 The *structural rules* for Logic are given in the following table. ▲

$\frac{}{A \vdash A} \text{Id}$	$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \text{Exch}$
$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{Contr}$	$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \text{Weak}$

Table 1.8: Structural Rules for Logic.

If we think of using proof rules *backwards* to reduce the task of proving a given sequent to various sub-tasks, then we see that the Contraction rule lets us *duplicate* premises, and the Weakening rule lets us *discard* them, while the Exchange rule merely lets us re-order them. The Identity axiom as given here is equivalent to the one with auxiliary premises given previously in the presence of Weakening.

The structural rules have clear categorical meanings in a category \mathcal{C} with products. Recalling the *diagonal transformation* $\Delta_A := \langle \text{id}_A, \text{id}_A \rangle$ and the *symmetry transformation* $s_{A,B} := \langle \pi_2, \pi_1 \rangle$, the meanings are as follows.

$$\begin{array}{ll}
 \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \text{Exch} & \frac{f : \Gamma \times A \times B \times \Delta \longrightarrow C}{f \circ (\text{id}_\Gamma \times s_{A,B} \times \text{id}_\Delta) : \Gamma \times B \times A \times \Delta \longrightarrow C} \\
 \\
 \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{Contr} & \frac{f : \Gamma \times A \times A \longrightarrow B}{f \circ (\text{id}_\Gamma \times \Delta_A) : \Gamma \times A \longrightarrow B} \\
 \\
 \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \text{Weak} & \frac{f : \Gamma \longrightarrow B}{f \circ \pi_1 : \Gamma \times A \longrightarrow B}
 \end{array}$$

In order to analyse Natural Deduction, Gentzen introduced **sequent calculi** based on **Left** and **Right** rules, instead of Introduction and Elimination rules. These kind of systems are more adequate for our discussion on linearity.

Definition 108 We define the *Gentzen sequent calculus* for \wedge, \supset as the proof system obtained by the structural rules (def. 107) and the rules in table 1.9 for connectives. ▲

Conjunction	Implication	Cut
$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge R$	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset R$	$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \text{Cut}$
$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$	$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \supset B, \Delta \vdash C} \supset L$	

Table 1.9: Gentzen Sequent Calculus for \wedge, \supset .

For example, the proof of \supset -transitivity is now given as follows.

$$\begin{array}{c}
\frac{\overline{A \vdash A} \text{ Id} \quad \overline{B \vdash B} \text{ Id}}{A, A \supset B \vdash B} \supset L \\
\frac{A \supset B, A \vdash B}{A \supset B, A \vdash B} \text{ Exch} \quad \overline{C \vdash C} \text{ Id} \\
\frac{\quad}{A \supset B, A, B \supset C \vdash C} \supset L \\
\frac{A \supset B, B \supset C, A \vdash C}{A \supset B, B \supset C, A \vdash C} \text{ Exch} \\
\frac{\quad}{A \supset B, B \supset C \vdash A \supset C} \supset R
\end{array}$$

Exercise 109 Show that the Gentzen-rules are admissible in Natural Deduction. Moreover, show that the Natural Deduction rules are admissible in the Gentzen sequent calculus.

The Cut rule allows the use of *lemmas* in proofs. It also yields a *dynamics of proofs* via *Cut Elimination*, that is, a dynamics of proof transformations towards the goal of eliminating the uses of the Cut rule in a proof, *i.e.* removing all lemmas and making the proof completely ‘explicit’, meaning Cut-free. Such transformations are always possible, as is shown in the following seminal result of Gentzen (*Hauptsatz*).

Fact 110 (Cut Elimination) *The Cut rule is admissible in the Gentzen sequent calculus without Cut.*

1.7.2 Linear Logic

In the presence of the structural rules, the Gentzen sequent calculus is entirely equivalent to the Natural Deduction system we studied earlier. Nevertheless:

What happens if we **drop** the Contraction and Weakening rules (but keep the Exchange rule)?

It turns out we can still make good sense of the resulting proofs, terms and categories, but now in the setting of a different, ‘resource-sensitive’ logic.

Definition 111 *Multiplicative Linear logic* is a variant of standard logic with *linear* logical connectives. The multiplicative connectives for conjunction and implication are \otimes and \multimap . Proof sequents are of the form $\Gamma \vdash A$, where Γ is now a *multiset*. The proof rules for \otimes, \multimap -Linear Logic, given in table 1.10, are the multiplicative versions of the Gentzen rules. \blacktriangle

Conjunction	Implication	Cut
$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes R$	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \multimap R$	$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \text{Cut}$
$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \otimes L$	$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \multimap L$	

Table 1.10: Rules for \otimes, \multimap -Linear Logic.

Multiplicativity here means the use of *disjoint* (*i.e.* non-overlapping) contexts. The use of multisets allows us to omit explicit use of the Exchange rule in our proof system.

Note that the given system satisfies Cut-elimination, and this leans heavily on the $\multimap L$ rule. We could have used instead the following rule,

$$\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

which is more intuitive computationally, but then cut-elimination would fail. Note, though, that:

$$\multimap L, \text{Cut}, \text{Id} \equiv \multimap E, \text{Cut}, \text{Id}.$$

This is shown as follows.

$$\frac{\frac{\Gamma \vdash A \quad \overline{A \multimap B \vdash A \multimap B}}{\Gamma, A \multimap B \vdash B} \text{Id} \multimap E \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \text{Cut} \quad \frac{\Gamma \vdash A \multimap B \quad \frac{\Delta \vdash A \quad \overline{B \vdash B}}{A \multimap B, \Delta \vdash B} \text{Id} \multimap L}{\Gamma, \Delta \vdash B} \text{Cut}$$

The resource-sensitive nature of Linear Logic is reflected in the following exercise.

Exercise 112 Can you construct proofs in Linear Logic of the following sequents? (Hint: Use the Cut Elimination property.)

- $A \vdash A \otimes A$
- $(A \otimes A) \multimap B \vdash A \multimap B$

- $\vdash A \multimap (B \multimap A)$

Related to linear logic is the *linear λ -calculus*, which is a linear version of the simply-typed λ -calculus.

Definition 113 The *linear λ -calculus* is defined as follows.

Type $\text{TY} \ni T, U ::= b \mid T \multimap U \mid T \otimes U$
 Term $\text{TE} \ni t, u ::= x \mid tu \mid \lambda x. t \mid t \otimes u \mid \text{let } z \text{ be } x \otimes y \text{ in } t$
 Typing context $\Gamma ::= \emptyset \mid x : T, \Gamma$

Terms are typed by use of the typing rules of table 1.11. Finally, the rules for β -reduction are:

$$\begin{aligned} (\lambda x. t)u &\longrightarrow_{\beta} t[u/x] \\ \text{let } t \otimes u \text{ be } x \otimes y \text{ in } v &\longrightarrow_{\beta} v[t/x, u/y]. \end{aligned}$$

▲

Note here that, again, $x : T, \Gamma$ stands for $\{x : T\} \cup \Gamma$ with x not appearing in Γ . Note also that Cut-free proofs *always yield terms in normal form*.

Variable / Cut	$\frac{}{x : T \vdash x : T}$	$\frac{\Gamma \vdash t : T \quad x : T, \Delta \vdash u : U}{\Gamma, \Delta \vdash u[t/x] : U}$
Linear Tensor	$\frac{\Gamma \vdash t : T \quad \Delta \vdash u : U}{\Gamma, \Delta \vdash t \otimes u : T \otimes U}$	$\frac{\Gamma, x : T, y : U \vdash v : V}{\Gamma, z : T \otimes U \vdash \text{let } z \text{ be } x \otimes y \text{ in } v : V}$
Linear Function	$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \multimap T}$	$\frac{\Gamma \vdash t : T \quad x : U, \Delta \vdash u : V}{\Gamma, f : T \multimap U, \Delta \vdash u[ft/x] : V}$

Table 1.11: Linear λ -calculus for \otimes, \multimap .

Term formation is now highly constrained by the form of the typing judgements. In particular,

$$x_1 : A_1, \dots, x_k : A_k \vdash t : A$$

now implies that each x_i occurs **exactly once** (free) in t .

Moreover, note that, for function application, instead of the rule on the LHS below, we could have used the more intuitive rule on the RHS.

$$\frac{\Gamma \vdash t : T \quad x : U, \Delta \vdash u : V}{\Gamma, f : T \multimap U, \Delta \vdash u[ft/x] : V} \qquad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}$$

As we did in the logic, we can show that the typing systems with one or the other rule are equivalent.

1.7.3 Linear Logic in Monoidal Categories

We proceed to give a categorical counterpart to linearity by providing a categorical interpretation of linear logic. Note that CCC's are no longer adequate for this task as they contain arrows

$$\Delta_A : A \longrightarrow A \times A, \quad \pi_1 : A \times B \longrightarrow A$$

which violate linearity. It turns out that the right setting is that of *symmetric monoidal closed categories*.

Definition 114 A *monoidal category* is a structure $(\mathcal{C}, \otimes, I, a, l, r)$ where:

- \mathcal{C} is a category,
- $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor (*tensor*),
- I is a distinguished object of \mathcal{C} (*unit*),
- a, l, r are natural isomorphisms (*structural isos*) with components:

$$a_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$$

$$l_A : I \otimes A \xrightarrow{\cong} A \quad r_A : A \otimes I \xrightarrow{\cong} A$$

such that $l_I = r_I : I \otimes I \rightarrow I$ and the following diagrams commute.

$$\begin{array}{ccccc}
 A \otimes (I \otimes B) & \xrightarrow{a} & (A \otimes I) \otimes B & & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \text{id} \otimes l & \nearrow r \otimes \text{id} & & \nearrow a & \searrow a \\
 A \otimes B & & A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \\
 & & \downarrow \text{id} \otimes a & & \uparrow a \otimes \text{id} \\
 & & A \otimes ((B \otimes C) \otimes D) & \xrightarrow{a} & (A \otimes (B \otimes C)) \otimes D
 \end{array}$$

▲

The monoidal diagrams ensure *coherence*, described by the slogan:

“... ‘all’ diagrams involving a, l and r must commute.”

Examples:

- Both products and coproducts give rise to monoidal structures—which are the common denominator between them. (But in addition, products

have *diagonals* and *projections*, and coproducts have *codiagonals* and *injections*.)

- $(\mathbb{N}, \leq, +, 0)$ is a monoidal category.
- **Rel**, the category of sets and relations, with cartesian product (which is *not* the categorical product).
- **Vect_k** with the tensor product.

Let us examine the example of **Rel** in some detail. We take \otimes to be the cartesian product, which is defined on relations $R : X \rightarrow X'$ and $S : Y \rightarrow Y'$ as follows.

$$\forall (x, y) \in X \times Y, (x', y') \in X' \times Y'. (x, y)R \otimes S(x', y') \iff xRx' \wedge ySy'.$$

It is not difficult to show that this is indeed a functor. Note that, in the case that R, S are *functions*, $R \otimes S$ is the same as $R \times S$ in **Set**. Moreover, we take each $a_{A,B,C}$ to be the associativity function for products (in **Set**), which is an iso in **Set** and hence also in **Rel**. Finally, we take I to be the one-element set, and l_A, r_A to be the projection functions: their relational converses are their inverses in **Rel**. The monoidal diagrams commute simply because they commute in **Set**.

Exercise 115 Verify that $(\mathbb{N}, \leq, +, 0)$ and **Vect_k** are monoidal categories.

Tensors and products

As we mentioned earlier, products are tensors with extra structure: natural diagonals and projections. This fact, which reflects *no-cloning* and *no-deleting* of Linear Logic, is shown as follows.

Proposition 116 *Let \mathcal{C} be a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$. \otimes induces a product structure iff there exist natural diagonals and projections, i.e. natural transformations given by arrows*

$$d_A : A \longrightarrow A \otimes A, \quad p_{A,B} : A \times B \longrightarrow A, \quad q_{A,B} : A \times B \longrightarrow B,$$

such that the following diagrams commute.

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & \downarrow d_A & \searrow \text{id}_A \\ A & A \otimes A & A \\ p_{A,A} \longleftarrow & & \longrightarrow q_{A,A} \end{array} \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{d_{A,B}} & (A \otimes B) \otimes (A \otimes B) \\ & \searrow \text{id}_{A \otimes B} & \downarrow p_{A,B} \otimes q_{A,B} \\ & & A \otimes B \end{array}$$

Proof: The “only if” direction is straightforward. For the converse, let \mathcal{C} be monoidal with natural projections and diagonals. Then, we take product

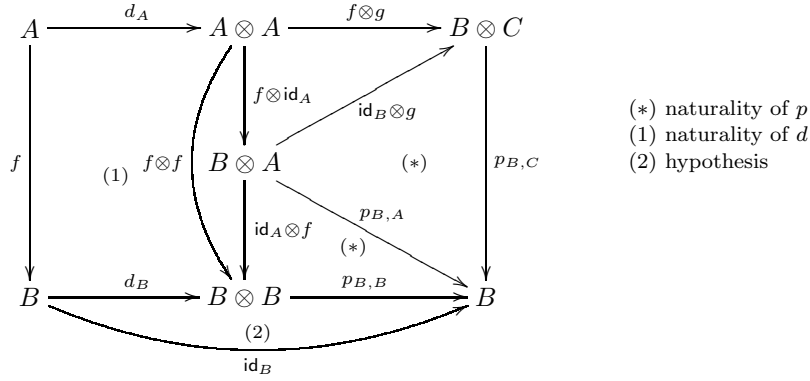
pairs to be pairs of the form

$$A \xleftarrow{p_{A,B}} A \otimes B \xrightarrow{q_{A,B}} B.$$

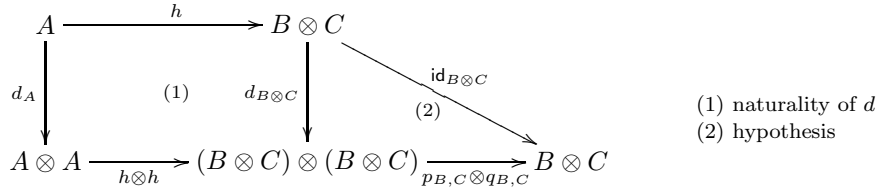
Moreover, for any pair of arrows $B \xleftarrow{f} A \xrightarrow{g} C$, define

$$\langle f, g \rangle := A \xrightarrow{d_A} A \otimes A \xrightarrow{f \otimes g} B \otimes C.$$

Then the product diagram commutes. For example:



For uniqueness, if $h : A \rightarrow B \otimes C$ then the following diagram commutes,



so $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$. ■

SMCC's

Linear Logic is interpreted in monoidal categories with two more pieces of structure: monoidal symmetry and closure. The former allows the Exchange rule to be interpreted, while the latter realises linear implication.

Definition 117 A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ with an additional natural isomorphism (*symmetry*),

$$s_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

such that $s_{B,A} = s_{A,B}^{-1}$ and the following diagrams commute.

$$\begin{array}{ccc}
 A \otimes I & \xrightarrow{s} & I \otimes A \\
 & \searrow r & \downarrow l \\
 & & A
 \end{array}
 \quad
 \begin{array}{ccccc}
 A \otimes (B \otimes C) & \xrightarrow{\text{id} \otimes s} & A \otimes (C \otimes B) & \xrightarrow{a} & (A \otimes C) \otimes B \\
 \downarrow a & & & & \downarrow s \otimes \text{id} \\
 (A \otimes B) \otimes C & \xrightarrow{s} & C \otimes (A \otimes B) & \xrightarrow{a} & (C \otimes A) \otimes B
 \end{array}$$

▲

Definition 118 A *symmetric monoidal closed category (SMCC)* is a symmetric monoidal category $(\mathcal{C}, \otimes, I, a, l, r, s)$ such that, for each object A , there is a couniversal arrow to the functor

$$- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}.$$

That is, for all pairs A, B , there is an object $A \multimap B$ and a morphism

$$\text{ev}_{A,B} : (A \multimap B) \otimes A \longrightarrow B$$

such that, for every morphism $f : C \otimes A \rightarrow B$, there is a *unique* morphism $\Lambda(f) : C \rightarrow (A \multimap B)$ such that

$$\text{ev}_{A,B} \circ (\Lambda(f) \otimes \text{id}_A) = f.$$

▲

Note that, although we use notation borrowed from CCC's (ev, Λ) , these are *different* structures! Examples of symmetric monoidal closed categories are **Rel**, **Vect_k**, and (*a fortiori*) cartesian closed categories.

Exercise 119 Show that **Rel** is a symmetric monoidal closed category.

Linear logic in SMCC's

Just as cartesian closed categories correspond to \wedge, \supset -logic (and simply-typed λ -calculus), so do symmetric monoidal closed categories correspond to \otimes, \multimap -logic (and linear λ -calculus).

So let \mathcal{C} be a symmetric monoidal closed category. The interpretation of a linear sequent

$$A_1, \dots, A_k \vdash A$$

will be a morphism

$$f : A_1 \otimes \dots \otimes A_k \longrightarrow A.$$

To be precise in our interpretation, we will again treat contexts as lists of formulas, and explicitly interpret the Exchange rule by:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \quad \frac{f : \Gamma \otimes A \otimes B \otimes \Delta \longrightarrow C}{f \circ (\text{id}_\Gamma \otimes s_{B,A} \otimes \text{id}_\Delta) : \Gamma \otimes B \otimes A \otimes \Delta \longrightarrow C}$$

The rest of the rules are translated as follows.

$\overline{A \vdash A}$	$\overline{\text{id}_A : A \longrightarrow A}$
$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$	$\frac{f : \Gamma \longrightarrow A \quad g : A \otimes \Delta \longrightarrow B}{g \circ (f \otimes \text{id}_\Delta) : \Gamma \otimes \Delta \longrightarrow B}$
$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$	$\frac{f : \Gamma \longrightarrow A \quad g : \Delta \longrightarrow B}{f \otimes g : \Gamma \otimes \Delta \longrightarrow A \otimes B}$
$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$	$\frac{f : (\Gamma \otimes A) \otimes B \longrightarrow C}{f \circ a_{\Gamma, A, B} : \Gamma \otimes (A \otimes B) \longrightarrow C}$
$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$	$\frac{f : \Gamma \otimes A \longrightarrow B}{\Lambda(f) : \Gamma \longrightarrow (A \multimap B)}$
$\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$	$\frac{f : \Gamma \longrightarrow (A \multimap B) \quad g : \Delta \longrightarrow A}{\text{ev}_{A,B} \circ (f \otimes g) : \Gamma \otimes \Delta \longrightarrow B}$

Table 1.12: Categorical Translation of \otimes, \multimap -Linear Logic.

Note that, because of coherence in monoidal categories, we will not be scholastic with associativity arrows a in our translations and will usually omit them. For the same reason, consecutive applications of tensor will be written without specifying associativity, e.g. $A_1 \otimes \cdots \otimes A_n$.

Exercise 120 Let \mathcal{C} be a symmetric monoidal closed category. Give the interpretation of the \multimap -left rule in \mathcal{C} :

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \multimap \text{L}$$

Exercise 121 Is it possible to translate \otimes, \multimap -logic into a CCC \mathcal{C} ? Is this in accordance with linearity of \otimes, \multimap -logic?

1.7.4 Beyond the Multiplicatives

Linear Logic has three ‘levels’ of connectives, each describing a different aspect of standard logic:

- The **multiplicatives**: e.g. \otimes, \multimap ,

- The **additives**: *additive conjunction* $\&$ and *disjunction* \oplus ,
- The **exponentials**, allowing controlled access to copying and discarding.

We focus on additive conjunction and the exponential “!”, which will allow us to recover the ‘expressive power’ of standard \wedge, \supset -logic.

Definition 122 The logical connective for **additive disjunction** is $\&$, and the related proof rules are the following.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \&R \quad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \&L \quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C} \&L$$

▲

So additive conjunction has proof rules that are identical to those of standard conjunction (\wedge). Note though that, since by linearity an argument of type $A \& B$ can only be used once, each use of a left rule for $\&$ makes a once-and-for-all *choice* of a projection. On the other hand, $A \otimes B$ represents a conjunction where *both* projections must be available.

Additive conjunction can be interpreted in any symmetric monoidal category with products, *i.e.* a category \mathcal{C} with structure (\otimes, \times) where \otimes is a symmetric monoidal tensor and \times is a product.

$$\frac{f : \Gamma \longrightarrow A \quad g : \Gamma \longrightarrow B}{\langle f, g \rangle : \Gamma \longrightarrow A \times B} \quad \frac{f : \Gamma \otimes A \longrightarrow C}{f \circ (\text{id} \otimes \pi_1) : \Gamma \otimes (A \times B) \longrightarrow C}$$

Moreover, we can extend the linear λ -calculus with term constructors for additive conjunction as follows.

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \& B} \quad \frac{\Gamma, x : A \vdash t : C}{\Gamma, z : A \& B \vdash \text{let } z = \langle x, - \rangle \text{ in } t : C} \quad \frac{\Gamma, B \vdash C}{\Gamma, z : A \& B \vdash \text{let } z = \langle -, y \rangle \text{ in } t : C}$$

The β -reduction rules related to these constructs are:

$$\begin{aligned} \text{let } \langle t, u \rangle = \langle x, - \rangle \text{ in } v &\longrightarrow_{\beta} v[t/x] \\ \text{let } \langle t, u \rangle = \langle -, y \rangle \text{ in } v &\longrightarrow_{\beta} v[u/y]. \end{aligned}$$

Finally, we can gain back the lost structural rules, in *disciplined* versions, by introducing an exponential *bang* operator $!$ which is a kind of *modality* enabling formulas to participate in structural rules.

Definition 123 The logical connective for **bang** is $!$, and the related proof rules are the following.

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} !L \quad \frac{! \Gamma \vdash A}{! \Gamma \vdash !A} !R \quad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \text{Weak} \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \text{Contr}$$

Note that $!\{A_1, \dots, A_n\} := !A_1, \dots, !A_n$. ▲

We can now see the discipline imposed on structural rules: in order for the rules to be applied, the participating formulas need to be tagged with a bang.

Interpreting standard Logic

We are now in position to recover the standard logical connectives \wedge, \supset within Linear Logic. If we interpret

$$\begin{aligned} A \supset B &:= !A \multimap B \\ A \wedge B &:= A \& B \end{aligned}$$

and each \wedge, \supset -sequent $\Gamma \vdash A$ as $!\Gamma \vdash A$, then each proof rule of the Gentzen system for \wedge, \supset is admissible in the proof system of Linear Logic for $\otimes, \multimap, \&, !$.

Note in particular that the interpretation

$$A \supset B := !A \multimap B$$

decomposes the fundamental notion of implication into finer notions—like ‘splitting the atom of logic’!

1.7.5 Exercises

1. Give proofs of the following sequents in Linear Logic.

- a. $\vdash A \multimap A$
- b. $A \multimap B, B \multimap C \vdash A \multimap C$
- c. $\vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)$
- d. $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$
- e. $A \otimes B \vdash B \otimes A$

For each of the proofs constructed give:

- the corresponding linear λ -term,
- its interpretation in **Rel**.

2. Consider a symmetric monoidal closed category \mathcal{C} .

- a. Suppose the sequents $\Gamma_1 \vdash A$, $\Gamma_2 \vdash B$ and $A, B, \Delta \vdash C$ are provable and let their interpretations (*i.e.* the interpretations of their proofs) in \mathcal{C} be $f_1 : \Gamma_1 \rightarrow A$, $f_2 : \Gamma_2 \rightarrow B$ and $g : A \otimes B \otimes \Delta \rightarrow C$ respectively. Find then the interpretations h_1, h_2 of the following proofs.

$$\frac{\frac{\frac{\vdots}{\Gamma_1 \vdash A} \quad \frac{\vdots}{\Gamma_2 \vdash B}}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \otimes R \quad \frac{\frac{\vdots}{A, B, \Delta \vdash C}}{A \otimes B, \Delta \vdash C} \otimes L}{\Gamma_1, \Gamma_2, \Delta \vdash C} \text{Cut} \quad \frac{\frac{\vdots}{\Gamma_2 \vdash B} \quad \frac{\frac{\frac{\vdots}{\Gamma_1 \vdash A} \quad \frac{\vdots}{A, B, \Delta \vdash C}}{\Gamma_1, B, \Delta \vdash C} \text{Cut}}{\Gamma_1, \Gamma_2, \Delta \vdash C} \text{Cut}$$

and show that $h_1 = h_2$.

- b. Suppose now \mathcal{C} has *also* binary products, given by \times . Given that the sequents $\Gamma \vdash A$, $\Gamma \vdash B$ and $A, \Delta \vdash C$ are provable, and that their interpretations in \mathcal{C} are $f_1 : \Gamma \rightarrow A$, $f_2 : \Gamma \rightarrow B$ and $g : A \otimes \Delta \rightarrow C$ respectively, find the interpretations h_1, h_2 of the following proofs.

$$\frac{\frac{\frac{\vdots}{\Gamma \vdash A} \quad \frac{\vdots}{\Gamma \vdash B}}{\Gamma \vdash A \& B} \& R \quad \frac{\frac{\vdots}{A, \Delta \vdash C}}{A \& B, \Delta \vdash C} \& L}{\Gamma, \Delta \vdash C} \text{Cut} \quad \frac{\frac{\vdots}{\Gamma \vdash A} \quad \frac{\vdots}{A, \Delta \vdash C}}{\Gamma, \Delta \vdash C} \text{Cut}$$

and show that $h_1 = h_2$.

3. Show that the condition $l_I = r_I$ in the definition of monoidal categories is redundant.
Moreover, show that the condition $\text{id}_A \otimes l_B = a_{A, I, B} \circ r_A \otimes \text{id}_B$ in the definition of symmetric monoidal categories is redundant.

1.8 Monads and Comonads

Recall that an adjunction is given by a triple $\langle F, G, \theta \rangle$, with $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ being functors, and θ a natural bijection between homsets. By composing the two functors we obtain endofunctors

$$G \circ F : \mathcal{C} \longrightarrow \mathcal{C}, \quad F \circ G : \mathcal{D} \longrightarrow \mathcal{D}.$$

These can be seen as *encapsulating the effect* of the adjunction inside their domain category. For example, if we consider the functors

$$\mathbf{MList} : \mathbf{Set} \longrightarrow \mathbf{Mon}, \quad U : \mathbf{Mon} \longrightarrow \mathbf{Set},$$

then $U \circ \mathbf{MList}$ encodes the free monoid construction inside **Set**.

The study of such endofunctors on their own right gave rise to the notions of *monad* and *comonad*, which we examine in this section.

1.8.1 Basics

Definition 124 A *monad* over a category \mathcal{C} is a triple (T, η, μ) where T is an endofunctor on \mathcal{C} and $\eta : \text{Id}_{\mathcal{C}} \rightarrow T$, $\mu : T^2 \rightarrow T$ are natural transformations such that the following diagrams commute. (Note that $T^2 := T \circ T$, etc.).

$$\begin{array}{ccc}
 T^3 A & \xrightarrow{\mu_{TA}} & T^2 A \\
 T\mu_A \downarrow & & \downarrow \mu_A \\
 T^2 A & \xrightarrow{\mu_A} & TA
 \end{array}
 \qquad
 \begin{array}{ccc}
 TA & \xrightarrow{\eta_{TA}} & T^2 A \\
 T\eta_A \downarrow & \searrow \text{id}_{TA} & \downarrow \mu_A \\
 T^2 A & \xrightarrow{\mu_A} & TA
 \end{array}
 \quad \blacktriangle$$

We call η the *unit* of the monad, and μ its *multiplication*; the whole terminology comes from monoids. Let us now proceed to some examples.

- Let \mathcal{C} be a category with coproducts and let E be an object in \mathcal{C} . We can define a monad (T, η, μ) of E -coproducts (computationally, E -exceptions) by taking $T : \mathcal{C} \rightarrow \mathcal{C}$ to be the functor $_ + E$, and η, μ as follows.

$$\begin{aligned}
 T &:= A \mapsto A + E, f \mapsto f + \text{id}_E \\
 \eta_A &:= A \xrightarrow{\text{in}_1} A + E \\
 \mu_A &:= (A + E) + E \xrightarrow{[\text{id}_{A+E}, \text{in}_2]} A + E
 \end{aligned}$$

As an injection, η is a natural transformation. For μ , we can use the properties of the coproduct. For $f : A \rightarrow B$,

$$\begin{aligned}
 Tf \circ \mu_A &= Tf \circ [\text{id}_{A+E}, \text{in}_2] = [Tf \circ \text{id}_{A+E}, Tf \circ \text{in}_2] = [Tf, Tf \circ \text{in}_2] \\
 &= [Tf, (f + \text{id}_E) \circ \text{in}_2] = [Tf, \text{in}_2] \\
 &= [\text{id}_{B+E} \circ Tf, \text{in}_2 \circ \text{id}_E] = [\text{id}_{B+E}, \text{in}_2] \circ (Tf + \text{id}_E) \\
 &= \mu_B \circ T^2 f.
 \end{aligned}$$

The monadic diagrams follow in a similar manner. For example,

$$\begin{aligned}
 \mu_A \circ \mu_{TA} &= \mu_A \circ [\text{id}_{TA+E}, \text{in}_2] = [\mu_A \circ \text{id}_{TA+E}, \mu_A \circ \text{in}_2] = [\mu_A, \mu_A \circ \text{in}_2] \\
 &= [\mu_A, [\text{id}_{A+E}, \text{in}_2] \circ \text{in}_2] = [\mu_A, \text{in}_2] \\
 &= [\text{id}_{A+E} \circ \mu_A, \text{in}_2 \circ \text{id}_E] = [\text{id}_{A+E}, \text{in}_2] \circ (\mu_A + \text{id}_E) \\
 &= \mu_A \circ T\mu_A.
 \end{aligned}$$

- Now let \mathcal{C} be a cartesian closed category and let ξ be some object in \mathcal{C} . We can define a monad of ξ -side-effects by taking T to be the functor $\xi \Rightarrow (_ \times \xi)$, and η, μ as follows.

$$\begin{aligned}
T &:= A \mapsto \xi \Rightarrow (A \times \xi), \quad f \mapsto \xi \Rightarrow (f \times \text{id}_\xi) \\
\eta_A &:= \Lambda(A \times \xi \xrightarrow{\text{id}_{A \times \xi}} A \times \xi) \\
\mu_A &:= \Lambda(T(TA) \times \xi \xrightarrow{\text{ev}_{\xi, TA \times \xi}} TA \times \xi \xrightarrow{\text{ev}_{\xi, A \times \xi}} A \times \xi)
\end{aligned}$$

Naturality of η, μ follows from naturality of Λ : for any $f : A \rightarrow A'$,

$$\begin{aligned}
Tf \circ \eta_A &= (\xi \Rightarrow f \times \text{id}_\xi) \circ \Lambda(\text{id}_{A \times \xi}) = \Lambda(f \times \text{id}_\xi \circ \text{id}_{A \times \xi}) \\
&= \Lambda(\text{id}_{A' \times \xi} \circ f \times \text{id}_\xi) = \Lambda(\text{id}_{A' \times \xi}) \circ f = \eta_{A'} \circ f, \\
\mu_{A'} \circ T^2f &= \Lambda(\text{ev}_{\xi, A' \times \xi} \circ \text{ev}_{\xi, TA' \times \xi}) \circ T^2f = \Lambda(\text{ev}_{\xi, A' \times \xi} \circ \text{ev}_{\xi, TA' \times \xi} \circ T^2f \times \text{id}_\xi) \\
&= \Lambda(\text{ev}_{\xi, A' \times \xi} \circ Tf \times \text{id}_\xi \circ \text{ev}_{\xi, TA \times \xi}) = \Lambda(f \times \text{id}_\xi \circ \text{ev}_{\xi, A \times \xi} \circ \text{ev}_{\xi, TA \times \xi}) \\
&= (\xi \Rightarrow f \times \text{id}_\xi) \circ \Lambda(\text{ev}_{\xi, A \times \xi} \circ \text{ev}_{\xi, TA \times \xi}) = Tf \circ \mu_A.
\end{aligned}$$

The monadic diagrams are shown in a similar manner.

- Our third example employs the functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$. In particular, we take $T := U \circ \mathbf{MList}$ and η, μ as follows.

$$\begin{aligned}
T &:= X \mapsto \bigcup_{n \in \omega} \{[x_1, \dots, x_n] \mid x_1, \dots, x_n \in X\}, \\
f &\mapsto ([x_1, \dots, x_n] \mapsto [f(x_1), \dots, f(x_n)]). \\
\eta_X &:= x \mapsto [x] \\
\mu_X &:= [[x_{11}, \dots, x_{1n_1}], \dots, [x_{k1}, \dots, x_{kn_k}]] \mapsto [x_{11}, \dots, x_{1n_1}, \dots, x_{k1}, \dots, x_{kn_k}]
\end{aligned}$$

Naturality of η, μ is obvious — besides, η is the unit of the corresponding adjunction. The monadic diagrams are also straightforward: they correspond to the following equalities of mappings (we use \mathbf{x} for x_1, \dots, x_n).

$$\begin{array}{ccc}
[[[\mathbf{x}_{11}], \dots, [\mathbf{x}_{1n_1}]], \dots, [[\mathbf{x}_{k1}], \dots, [\mathbf{x}_{kn_k}]]] & \xrightarrow{\mu} & [[\mathbf{x}_{11}], \dots, [\mathbf{x}_{1n_1}], \dots, [\mathbf{x}_{k1}], \dots, [\mathbf{x}_{kn_k}]] \\
\downarrow T\mu & & \downarrow \mu \\
[[\mathbf{x}_{11}, \dots, \mathbf{x}_{1n_1}], \dots, [\mathbf{x}_{k1}, \dots, \mathbf{x}_{kn_k}]] & \xrightarrow{\mu} & [\mathbf{x}_{11}, \dots, \mathbf{x}_{1n_1}, \dots, \mathbf{x}_{k1}, \dots, \mathbf{x}_{kn_k}]
\end{array}$$

$$\begin{array}{ccc}
[x_1, \dots, x_n] & \xrightarrow{\eta} & [[x_1, \dots, x_n]] \\
\downarrow T\eta & \searrow \text{id} & \downarrow \mu \\
[[x_1], \dots, [x_n]] & \xrightarrow{\mu} & [x_1, \dots, x_n]
\end{array}$$

Exercise 125 Show that the E -coproduct monad and the ξ -side-effect monads are indeed monads.

Our discussion on monads can be dualised, leading us to *comonads*.

Definition 126 A *comonad* over a category \mathcal{C} is a triple (Q, ε, δ) where Q is an endofunctor on \mathcal{C} and $\varepsilon : Q \rightarrow \text{Id}_{\mathcal{C}}$, $\delta : Q \rightarrow Q^2$ are natural transformations such that the following diagrams commute.

$$\begin{array}{ccc}
 QA & \xrightarrow{\delta_A} & Q^2A \\
 \delta_A \downarrow & & \downarrow \delta_{QA} \\
 Q^2A & \xrightarrow{Q\delta_A} & Q^3A
 \end{array}
 \qquad
 \begin{array}{ccc}
 QA & \xrightarrow{\delta_A} & Q^2A \\
 \delta_A \downarrow & \searrow \text{id}_{QA} & \downarrow \varepsilon_{QA} \\
 Q^2A & \xrightarrow{Q\varepsilon_A} & QA
 \end{array}$$

▲

ε is the *counit* of the comonad, and δ its *comultiplication*. Two of our examples from monads dualise to comonads.

- If \mathcal{C} has finite products then, for any object S , we can define the S -product comonad with functor $Q := S \times _$.
- We can form a comonad on **Mon** with functor $Q := \text{MList} \circ U$ (and counit that of the corresponding adjunction).

Exercise 127 Give an explicit description of the comonad on **Mon** with functor $Q := \text{MList} \circ U$ described above. Verify it is a comonad.

1.8.2 (Co)Monads of an Adjunction

In the previous section, we saw that an adjunction between **Mon** and **Set** yielded a monad on **Set** (and a comonad on **Mon**), with its unit being the unit of the adjunction. We now show that this observation generalises to any adjunction. Recall that an adjunction is specified by:

- a pair of functors $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$,
- for each $A \in \text{Ob}(\mathcal{C})$, $B \in \text{Ob}(\mathcal{D})$, a bijection $\theta_{A,B} : \mathcal{C}(A, GB) \cong \mathcal{D}(FA, B)$ natural in A, B .

For such an adjunction we build a monad on \mathcal{C} : the functor of the monad is simply $T := G \circ F$, and unit and multiplication are defined by setting

$$\begin{aligned}
 \eta_A : A &\longrightarrow GFA := \theta_{A,FA}^{-1}(\text{id}_{FA}), \\
 \mu_A : GFGFA &\longrightarrow GFA := G(\theta_{GFA,FA}(\text{id}_{GFA})).
 \end{aligned}$$

Observe that η is the unit of the adjunction.

Proposition 128 *Let (F, G, η) be an adjunction. Then the triple (T, η, μ) defined above is a monad on \mathcal{C} .*

Proof: Recall that naturality of θ means concretely that, for any $f : A \rightarrow GB$, $g : A' \rightarrow A$ and $h : B \rightarrow B'$,

$$\theta_{A',B'}(Gh \circ f \circ g) = h \circ \theta_{A,B}(f) \circ Fg.$$

η is the unit of the adjunction and hence natural. We show naturality of μ :

$$\begin{aligned} GF GF f \circ \mu_B &= G\theta_{GFB,FB}(\text{id}_{GFB}) \circ GF GF f = G(\theta_{GFB,FB}(\text{id}_{GFB}) \circ FGF f) \\ &\stackrel{\text{nat.}\theta}{=} G\theta_{GFA,FB}(\text{id}_{GFB} \circ GF f) = G\theta_{GFA,FB}(GF f \circ \text{id}_{GFA}) \\ &\stackrel{\text{nat.}\theta}{=} G(F f \circ \theta_{GFA,FA}(\text{id}_{GFA})) = GF f \circ \mu_A. \end{aligned}$$

The monoidal condition for μ also follows from naturality of θ :

$$\begin{aligned} \mu_A \circ \mu_{GFA} &= G(\theta(\text{id}_{GFA}) \circ \theta(\text{id}_{GF GFA})) \stackrel{\text{nat}}{=} G\theta(G\theta(\text{id}_{GFA}) \circ \text{id}_{GF GFA}) \\ &= G\theta(\text{id}_{GFA} \circ G\theta(\text{id}_{GFA})) \stackrel{\text{nat}}{=} G(\theta(\text{id}_{GFA}) \circ FG\theta(\text{id}_{GFA})) \\ &= \mu_A \circ GF \mu_A. \end{aligned}$$

Finally, for the η - μ conditions we also use the universality diagram for η and the uniqueness property (in equational form).

$$\begin{aligned} \mu_A \circ \eta_{GFA} &= G\theta_{GFA,FA}(\text{id}_{GFA}) \circ \eta_{GFA} = \text{id}_{GFA}, \\ \mu_A \circ GF \eta_{GFA} &= G\theta_{GFA,FA}(\text{id}_{GFA}) \circ GF \eta_{GFA} = G(\theta_{GFA,FA}(\text{id}_{GFA}) \circ F\eta_{GFA}) \\ &\stackrel{\text{nat}}{=} G\theta(\text{id}_{GFA} \circ \eta_{GFA}) = G\theta(G\text{id}_{FA} \circ \eta_{GFA}) = G\text{id}_{FA} = GF A. \end{aligned}$$

■

Hence, every adjunction gives rise to a monad. It turns out that the converse is also true: *every* monad is described by means of an adjunction in this way. In particular, there are two canonical constructions of adjunctions from a given monad: the *Kleisli construction*, and the *Eilenberg-Moore construction*. These are in a sense minimal and maximal solutions to describing a monad via an adjunction. We describe the former one in the next section.

Finally, note that — because of the symmetric definition of adjunctions — the whole discussion can be dualised to comonads. That is, every adjunction gives rise to a comonad with counit that of the adjunction, and also every comonad can be derived from an adjunction in this manner.

1.8.3 The Kleisli Construction

The Kleisli construction starts from a monad (T, η, μ) on a category \mathcal{C} and builds a category \mathcal{C}_T of T -computations, as follows.

Definition 129 Let (T, η, μ) be a monad on a category \mathcal{C} . Construct the **Kleisli category** \mathcal{C}_T by taking the same objects as \mathcal{C} , and by including an arrow $f.T : A \rightarrow B$ in \mathcal{C}_T for each $f : A \rightarrow TB$ in \mathcal{C} . That is,

$$\begin{aligned} Ob(\mathcal{C}_T) &:= Ob(\mathcal{C}), \\ \mathcal{C}_T(A, B) &:= \{f.T \mid f \in \mathcal{C}(A, TB)\}. \end{aligned}$$

The identity arrow for A in \mathcal{C}_T is $\eta_{A.T}$, while the composite of $f.T : A \rightarrow B$ and $g.T : B \rightarrow C$ is $h.T$, where:

$$h := A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC.$$

▲

The conditions for \mathcal{C}_T being a category follow from the monadic conditions. For composition with identity, for any $f : A \rightarrow TB$,

$$\begin{aligned} f.T \circ \eta_{A.T} &= (\mu_B \circ Tf \circ \eta_A).T = (\mu_B \circ \eta_B \circ f).T = f.T, \\ \eta_{B.T} \circ f.T &= (\mu_B \circ T\eta_B \circ f).T = f.T. \end{aligned}$$

For associativity of composition, for any $f : A \rightarrow TB$, $g : B \rightarrow TC$ and $h : C \rightarrow TD$,

$$\begin{aligned} (h.T \circ g.T) \circ f.T &= (\mu_D \circ Th \circ g).T \circ f.T = (\mu_D \circ T(\mu_D \circ Th \circ g) \circ f).T \\ &= (\mu_D \circ T\mu_D \circ T^2h \circ Tg \circ f).T = (\mu_D \circ \mu_{TD} \circ T^2h \circ Tg \circ f).T \\ &= (\mu_D \circ Th \circ \mu_C \circ Tg \circ f).T = h.T \circ (g.T \circ f.T). \end{aligned}$$

Let us now proceed to build the adjunction between \mathcal{C} and \mathcal{C}_T that will eventually give us back the monad T . Construct the functors $F : \mathcal{C} \rightarrow \mathcal{C}_T$ and $G : \mathcal{C}_T \rightarrow \mathcal{C}$ as follows.

$$\begin{aligned} F &:= A \mapsto A, (f : A \rightarrow B) \mapsto ((\eta_B \circ f).T : A \rightarrow B), \\ G &:= A \mapsto TA, (f.T : A \rightarrow B) \mapsto (\mu_B \circ Tf : TA \rightarrow TB). \end{aligned}$$

Functoriality of F, G follows from the monad laws and the definition of \mathcal{C}_T . Moreover, for each $A, B \in Ob(\mathcal{C})$, construct the following bijection of arrows.

$$\theta_{A,B} : \mathcal{C}(A, TB) \xrightarrow{\cong} \mathcal{C}_T(A, B) := f \mapsto f.T$$

To establish that (F, G, θ) is an adjunction we need only show that θ is natural in A, B . So take $f : A \rightarrow TB$, $g : A' \rightarrow A$ and $h.T : B \rightarrow B'$. We then have:

$$\begin{aligned} \theta_{A',B'}(G(h.T) \circ f \circ g) &= \theta_{A',B'}(\mu_{B'} \circ Th \circ f \circ g) = (\mu_{B'} \circ Th \circ f \circ g).T \\ &= h.T \circ (f \circ g).T = h.T \circ (\mu_B \circ Tf \circ \eta_A \circ g).T \\ &= h.T \circ f.T \circ (\eta_A \circ g).T = h.T \circ \theta_{A,B}(f) \circ Fg. \end{aligned}$$

The final step in this section is to verify that the monad (T', η', μ') arising from this adjunction is the one we started from. The construction of T' follows the recipe given in the previous section, that is:

- $T' : \mathcal{C} \rightarrow \mathcal{C} := G \circ F$. Thus, T' maps each object A to TA , and each arrow $f : A \rightarrow B$ to $\mu_B \circ T\eta_A \circ Tf = Tf$.
- $\eta'_A : A \rightarrow TA := \theta_{A, FA}^{-1}(\text{id}_{FA}^{(\mathcal{C}_T)}) = \theta^{-1}(\eta_{A.T}) = \eta_A$.
- $\mu'_A : T^2A \rightarrow TA := G\theta_{GFA, FA}(\text{id}_{GFA}^{(\mathcal{C})}) = G\theta(\text{id}_{TA}) = \mu_A \circ T\text{id}_{TA} = \mu_A$.

Thus, we have indeed obtained the initial (T, η, μ) .

The Kleisli construction on a comonad

Dually to the Kleisli category of a monad we can construct the Kleisli category of a comonad⁵ — and reobtain the comonad through an adjunction between the Kleisli category and the original one. Specifically, given a category \mathcal{C} and a comonad (Q, ε, δ) on \mathcal{C} , we define the category \mathcal{C}_Q as follows.

$$\begin{aligned} \text{Ob}(\mathcal{C}_Q) &:= \text{Ob}(\mathcal{C}) \\ \mathcal{C}_Q(A, B) &:= \{f.Q \mid f \in \mathcal{C}(QA, B)\} \\ \text{id}_A^{(\mathcal{C}_Q)} &:= \varepsilon_{A.Q} \\ g.Q \circ f.Q &:= (g \circ Qf \circ \delta_A).Q \end{aligned}$$

The Kleisli category of a comonad will be of use in the next sections, where comonads will be considered for modelling *bang* of Linear Logic. We end this section by showing a result that will be of use then.

Proposition 130 *Let \mathcal{C} be a category and (Q, ε, δ) be a comonad on \mathcal{C} . If \mathcal{C} has binary products then so does \mathcal{C}_Q .*

Proof: Let A, B be objects in $\mathcal{C}, \mathcal{C}_Q$. We claim that their product in \mathcal{C}_Q is given by $(A \times B, p_1, p_2)$, where

$$p_1 := (Q(A \times B) \xrightarrow{\varepsilon} A \times B \xrightarrow{\pi_1} A).Q$$

and similarly for p_2 . Now, for each $f.Q : C \rightarrow A$ and $g.Q : C \rightarrow B$, setting $\langle f.Q, g.Q \rangle := \langle f, g \rangle.Q$ we have:

$$p_1 \circ \langle f.Q, g.Q \rangle = (\pi_1 \circ \varepsilon \circ Q\langle f, g \rangle \circ \delta).Q = (\pi_1 \circ \langle f, g \rangle \circ \varepsilon \circ \delta).Q = f.Q,$$

and similarly $p_2 \circ \langle f.Q, g.Q \rangle = g.Q$. Finally, for any $h.Q : C \rightarrow A \times B$,

⁵ In some texts, this is called a *coKleisli* category.

Hence, we need to impose (a coherent) distributivity of the tensor—either binary (\otimes) or nullary (I)—over the comonad Q . This can be formalised by stipulating that Q be a *symmetric monoidal* endofunctor.

Definition 133 Let $(\mathcal{C}, \otimes, I, a, l, r, s)$ and $(\mathcal{C}', \otimes', I', a', l', r', s')$ be symmetric monoidal categories. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is called **symmetric monoidal** if there exist:

- a morphism $m_0 : I' \rightarrow F(I)$,
- a natural transformation $m_2 : F(-) \otimes' F(-) \rightarrow F(- \otimes -)$,

such that the following diagrams commute.

$$\begin{array}{ccc}
 FA \otimes' (FB \otimes' FC) & \xrightarrow{\text{id} \otimes' m_2} & FA \otimes' F(B \otimes C) \xrightarrow{m_2} F(A \otimes (B \otimes C)) \\
 \downarrow a' & & \downarrow Fa \\
 (FA \otimes' FB) \otimes' FC & \xrightarrow{m_2 \otimes' \text{id}} & F(A \otimes B) \otimes' FC \xrightarrow{m_2} F((A \otimes B) \otimes C)
 \end{array}$$

$$\begin{array}{ccc}
 FA \otimes' I' & \xrightarrow{\text{id} \otimes' m_0} & FA \otimes' FI \\
 \downarrow r' & & \downarrow m_2 \\
 FA & \xleftarrow{Fr} & F(A \otimes I)
 \end{array}
 \quad
 \begin{array}{ccc}
 FA \otimes' FB & \xrightarrow{m_2} & F(A \otimes B) \\
 \downarrow s' & & \downarrow Fs \\
 FB \otimes' FA & \xrightarrow{m_2} & F(B \otimes A)
 \end{array}$$

We may write such an F as (F, m) . Moreover, if $(F, m), (G, n) : \mathcal{C} \rightarrow \mathcal{C}'$ are (symmetric) monoidal functors then a natural transformation $\phi : F \rightarrow G$ is called **monoidal** whenever the following diagrams commute.

$$\begin{array}{ccc}
 I' & \xrightarrow{m_0} & FI \\
 & \searrow n_0 & \downarrow \phi \\
 & & GI
 \end{array}
 \quad
 \begin{array}{ccc}
 FA \otimes' FB & \xrightarrow{m_2} & F(A \otimes B) \\
 \downarrow \phi \otimes' \phi & & \downarrow \phi \\
 GA \otimes' GB & \xrightarrow{n_2} & G(A \otimes B)
 \end{array}
 \quad \blacktriangle$$

For example, the identity functor is symmetric monoidal. Moreover, if F and G are symmetric monoidal functors then so is $G \circ F$. Other examples are the following.

- The constant endofunctor K_I , which maps each object to I and each arrow to id_I , is symmetric monoidal with structure maps:

$$m_0 : I \longrightarrow I := \text{id}_I, \quad m_2 : I \otimes I \longrightarrow I := r_I.$$

- The endofunctor $\otimes \circ \langle \text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}} \rangle$, which maps each object A to $A \otimes A$ and each arrow f to $f \otimes f$, is symmetric monoidal with:

$$m_0 : I \longrightarrow I \otimes I := r_I^{-1}, \quad m_2 := (A \otimes A) \otimes (B \otimes B) \longrightarrow (A \otimes B) \otimes (A \otimes B),$$

the latter given by use of structural transformations.

Exercise 134 Verify that if $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{E}$ are symmetric monoidal functors then so is $G \circ F$.

Definition 135 A comonad (Q, ε, δ) on a SMCC \mathcal{C} is called a **monoidal comonad** if Q is a symmetric monoidal functor, say (Q, m) , and ε, δ are monoidal natural transformations. We write Q as $(Q, \varepsilon, \delta, m)$. \blacktriangle

Now let us assume \mathcal{C} is a SMCC and $(Q, \varepsilon, \delta, m)$ is a monoidal comonad on \mathcal{C} . The coherence of m_2 with a , expressed by the first diagram of symmetric monoidal functors, allows us to generalise m_0 and m_2 to arbitrary arities and assume arrows:

$$m_n : QA_1 \otimes \cdots \otimes QA_n \longrightarrow Q(A_1 \otimes \cdots \otimes A_n).$$

We can give the interpretation of the Right rule for bang as follows.

$$\frac{f : QB_1 \otimes \cdots \otimes QB_n \longrightarrow A}{Qf \circ m_n \circ (\delta_{B_1} \otimes \cdots \otimes \delta_{B_n}) : QB_1 \otimes \cdots \otimes QB_n \longrightarrow QA}$$

Contraction and Weakening

Our discussion on the categorical modelling of linear exponentials has only touched the issues of Right and Left rules. However, we also need adequate structure for translating Contraction and Weakening.

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \text{ Contr} \quad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \text{ Weak}$$

For these rules we can use appropriate (monoidal) natural transformations. For Contraction, we stipulate a transformation with components $d_A : QA \rightarrow QA \otimes QA$, *i.e.*

$$d : Q \longrightarrow \otimes \circ \langle Q, Q \rangle.$$

For Weakening, a transformation with components $e_A : QA \rightarrow I$, *i.e.*

$$e : Q \longrightarrow K_I.$$

Although the above allow the categorical interpretation of the proof-rules, they do not necessarily preserve the intended proof-transformations. For that, we need to impose some further coherence conditions, which are epitomised in the following notion.

Definition 136 Let \mathcal{C} be a SMCC. A monoidal comonad $(Q, \varepsilon, \delta, m)$ on \mathcal{C} is called a **linear exponential comonad** if there exist monoidal natural transformations

$$d : Q \longrightarrow \otimes \circ \langle Q, Q \rangle, \quad e : Q \longrightarrow K_I,$$

such that:

- (a) for each object A , the triple (QA, d_A, e_A) is a *commutative comonoid* in \mathcal{C} , i.e. the following diagrams commute,

$$\begin{array}{ccc} QA & \xrightarrow{d_A} & QA \otimes QA \\ \uparrow l_{QA} & \searrow d_A & \downarrow s_{QA, QA} \\ I \otimes QA & \xleftarrow{e_A \otimes \text{id}_{QA}} & QA \otimes QA \end{array} \quad \begin{array}{ccc} QA & \xrightarrow{d_A} & QA \otimes QA \\ \downarrow d_A & & \downarrow d_A \otimes \text{id}_{QA} \\ QA \otimes QA & \xrightarrow{a \circ (\text{id}_{QA} \otimes d_A)} & (QA \otimes QA) \otimes QA \end{array}$$

- (b) for each object A , the following diagrams commute.

$$\begin{array}{ccc} QA & \xrightarrow{\delta_A} & Q^2A \\ e_A \downarrow & & \downarrow Qe_A \\ I & \xrightarrow{m_0} & QI \end{array} \quad \begin{array}{ccc} QA & \xrightarrow{\delta_A} & Q^2A \\ d_A \downarrow & & \downarrow d_A \\ QA \otimes QA & \xrightarrow{\delta_A \otimes \delta_A} & Q^2A \otimes Q^2A \end{array} \quad \begin{array}{ccc} & & Qd_A \\ & & \searrow \\ & & Q(QA \otimes QA) \end{array}$$

$$\begin{array}{ccc} QA & \xrightarrow{\delta_A} & Q^2A \\ e_A \searrow & & \downarrow e_{QA} \\ & & I \end{array} \quad \begin{array}{ccc} QA & \xrightarrow{\delta_A} & Q^2A \\ d_A \downarrow & & \downarrow d_{QA} \\ QA \otimes QA & \xrightarrow{\delta_A \otimes \delta_A} & Q^2A \otimes Q^2A \end{array}$$

We write Q as $(Q, \varepsilon, \delta, m, d, e)$. ▲

Exercise 137 Express what it means concretely for d, e to be monoidal natural transformations.

Exercise 138 Give the categorical interpretation of Contraction and Weakening in a SMCC \mathcal{C} with a linear exponential comonad.

1.8.4.1 Including Products

We now consider the fragment of Linear Logic which includes all four linear connectives we have seen thus far, i.e. $\otimes \multimap !\&$, and their respective proof rules (see definitions 122, 123). The categorical modelling of $(\otimes \multimap !\&)$ -Linear Logic requires:

- a symmetric monoidal closed category \mathcal{C} ,
- a linear exponential comonad $(Q, \varepsilon, \delta, m, d, e)$ on \mathcal{C} ,
- finite products in \mathcal{C} .

The above structure is adequate for modelling the proof rules as we have seen previously. Moreover, it provides rich structure for the Kleisli category \mathcal{C}_Q . The next result and its proof demonstrate categorically the ‘interpretation’ of ordinary logic within Linear Logic given by:

$$A \Rightarrow B \equiv !A \multimap B.$$

Proposition 139 *Let \mathcal{C} be a SMCC with finite products and let $(Q, \varepsilon, \delta, m, d, e)$ be a linear exponential comonad on \mathcal{C} . Then:*

- (a) *The Kleisli category \mathcal{C}_Q has finite products.*
- (b) *There exists an isomorphism $i : Q\mathbf{1} \rightarrow I$ and a natural isomorphism $j : Q(- \times -) \rightarrow Q(-) \otimes Q(-)$.*
- (c) *\mathcal{C}_Q is cartesian closed, with the exponential of objects B, C being $QB \multimap C$.*

Proof: Part (a) has been shown previously (proposition 130, exercise 131), and part (b) is left as exercise. For (c), we have the following isomorphisms:

$$\begin{aligned} \mathcal{C}_Q(A \times B, C) &= \mathcal{C}(Q(A \times B), C) && \text{definition of } \mathcal{C}_Q \\ &\cong \mathcal{C}(QA \otimes QB, C) && \text{part (b)} \\ &\cong \mathcal{C}(QA, QB \multimap C) && \text{monoidal closure of } \mathcal{C} \\ &= \mathcal{C}_Q(A, QB \multimap C) && \text{defn of } \mathcal{C}_Q. \end{aligned}$$

Concretely, we obtain $\theta_A : \mathcal{C}_Q(A \times B, C) \xrightarrow{\cong} \mathcal{C}_Q(A, QB \multimap C)$ by:

$$\begin{aligned} \theta_A &:= (f.Q : A \times B \rightarrow C) \mapsto (A(f \circ j_{A,B}^{-1})).Q \\ \theta_A^{-1} &:= (g.Q : A \rightarrow QB \multimap C) \mapsto (A^{-1}(g) \circ j_{A,B}).Q. \end{aligned}$$

Clearly, θ_A is a bijection. In order to establish couniversality of the exponential, we need also show naturality in A (see exercise 76). So take $f.Q : A \times B \rightarrow C$ and $h.Q : A' \rightarrow A$. Note first that the following commutes.

$$\begin{array}{ccc} Q(A \times B) & \xrightarrow{\delta} & Q^2(A \times B) \xrightarrow{Q\langle Q\pi_1, Q\pi_2 \rangle} Q(QA \times QB) \\ \downarrow j & & \downarrow j \\ QA \otimes QB & \xrightarrow{\delta \otimes \delta} & Q^2A \otimes Q^2B \end{array} \quad (*)$$

Note also that, for any $h_{i,Q} : A'_i \rightarrow A_i$ in \mathcal{C}_Q , $i = 1, 2$, we have:

$$h_{1,Q} \times h_{2,Q} := (Q(A'_1 \times A'_2) \xrightarrow{\langle Q\pi_1, Q\pi_2 \rangle} QA'_1 \times QA'_2 \xrightarrow{h_1 \times h_2} A_1 \times A_2).Q$$

Thus, noting that $\text{id}_B^{(\mathcal{C}_Q)} = \varepsilon_{B,Q}$,

$$\begin{aligned}
\theta_{A'}(f.Q \circ h.Q \times \text{id}_B^{(C_Q)}) &= (\Lambda(f \circ Q(h \times \varepsilon \circ \langle Q\pi_1, Q\pi_2 \rangle) \circ \delta \circ j^{-1}))_{.Q} \\
&= (\Lambda(f \circ Q(h \times \varepsilon) \circ Q\langle Q\pi_1, Q\pi_2 \rangle \circ \delta \circ j^{-1}))_{.Q} \\
&\stackrel{(*)}{=} (\Lambda(f \circ Q(h \times \varepsilon) \circ j^{-1} \circ \delta \otimes \delta))_{.Q} \\
&= (\Lambda(f \circ j^{-1} \circ Qh \otimes Q\varepsilon \circ \delta \otimes \delta))_{.Q} \\
&= (\Lambda(f \circ j^{-1} \circ (Qh \circ \delta) \otimes \text{id}))_{.Q} \\
&= (\Lambda(f \circ j^{-1} \circ Qh \circ \delta))_{.Q} = \theta_A(f.Q) \circ h.Q
\end{aligned}$$

as required. ■

Exercise 140 Show part (b) of proposition 139. For the defined j , show commutativity of $(*)$.

1.8.5 Exercises

1. We say that a category \mathcal{C} is *well-pointed* if it contains a terminal object $\mathbf{1}$ and, for any pair of arrows $f, g : A \rightarrow B$,

$$f \neq g \implies \exists h : \mathbf{1} \rightarrow A. f \circ h \neq g \circ h.$$

Let now \mathcal{C} be a well-pointed category with a terminal object $\mathbf{1}$ and binary coproducts, and consider the functor $G : \mathcal{C} \rightarrow \mathcal{C}$ given by:

$$G := A \mapsto A + \mathbf{1}, f \mapsto f + \text{id}_{\mathbf{1}}.$$

If $\mathcal{C}(\mathbf{1}, \mathbf{1} + \mathbf{1}) = \{\text{in}_1, \text{in}_2\}$ with $\text{in}_1 \neq \text{in}_2$, show that if (G, η, μ) is a monad on \mathcal{C} then, for each object A :

$$\eta_A = A \xrightarrow{\text{in}_1} A + \mathbf{1}, \quad \mu_A = (A + \mathbf{1}) + \mathbf{1} \xrightarrow{[\text{id}_{A+\mathbf{1}}, \text{in}_2]} A + \mathbf{1}.$$

2. Let \mathcal{C} be a SMCC and let (Q, ε, δ) be a comonad on \mathcal{C} .
 - (a) Suppose that the sequents $\hat{\mathbf{!}}A \vdash B$ and $\hat{\mathbf{!}}B \vdash C$ are provable and let $f : QA \rightarrow B$ and $g : QB \rightarrow C$ be their interpretations (*i.e.* the interpretations of their proofs) in \mathcal{C} . Find the interpretations of the sequent $\hat{\mathbf{!}}A \vdash \hat{\mathbf{!}}C$ which correspond to each of the following proofs and show that the two interpretations are equal.

$$\begin{array}{c}
\frac{\frac{\vdots}{\hat{!}A \vdash \hat{!}B} \hat{!}R \quad \frac{\frac{\vdots}{\hat{!}B \vdash \hat{!}C} \hat{!}R}{\hat{!}B \vdash \hat{!}C} \hat{!}R}{\hat{!}A \vdash \hat{!}C} \text{Cut}
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\vdots}{\hat{!}A \vdash \hat{!}B} \hat{!}R \quad \frac{\frac{\vdots}{\hat{!}B \vdash \hat{!}C} \hat{!}R}{\hat{!}A \vdash \hat{!}C} \hat{!}R}{\hat{!}A \vdash \hat{!}C} \text{Cut}
\end{array}$$

- (b) Find the interpretations in \mathcal{C} of the following proofs; are the interpretations equal?

$$\begin{array}{c}
\frac{\frac{\hat{!}A \vdash \hat{!}A}{\hat{!}A \vdash \hat{!}A} \text{Id}}{\hat{!}A \vdash \hat{!}A} \hat{!}L
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\hat{!}A \vdash \hat{!}A}{\hat{!}A \vdash \hat{!}A} \text{Id}}{\hat{!}A \vdash \hat{!}A} \hat{!}R
\end{array}$$

3. Show that a symmetric monoidal category \mathcal{C} has finite products (given by \otimes, I , etc.) iff there are monoidal natural transformations

$$d : \text{Id}_{\mathcal{C}} \longrightarrow \otimes \circ \langle \text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}} \rangle, \quad e : \text{Id}_{\mathcal{C}} \longrightarrow K_I,$$

such that the following diagram commutes, for any $A \in \text{Ob}(\mathcal{C})$.

$$\begin{array}{ccc}
A & \xleftarrow{r_A} & A \otimes I \\
\uparrow l_A & \searrow d_A & \uparrow e_A \otimes \text{id}_A \\
I \otimes A & \xleftarrow{\text{id}_A \otimes e_A} & A \otimes A
\end{array}$$

1.9 Review of Sets, Functions and Relations

Our aim in this Appendix is to provide a brief review of notions we will assume in the notes. If the first paragraph is not familiar to you, you will need to acquire more background before being ready to read the notes.

Cartesian products, relations and functions

Given sets X and Y , their cartesian product is

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}.$$

A *relation* R from X to Y , written $R : X \rightarrow Y$, is a subset $R \subseteq X \times Y$. Given such a relation, we write $(x, y) \in R$, or equivalently $R(x, y)$. We compose relations as follows: if $R : X \rightarrow Y$ and $S : Y \rightarrow Z$, then for all $x \in X$ and $z \in Z$:

$$R; S(x, z) \equiv \exists y \in Y. R(x, y) \wedge S(y, z).$$

A relation $f : X \rightarrow Y$ is a *function* if it satisfies the following two properties:

- (single-valuedness): if $(x, y) \in f$ and $(x, y') \in f$, then $y = y'$.
- (totality): for all $x \in X$, for some $y \in Y$, $(x, y) \in f$.

If f is a function, we write $f(x) = y$ or $f : x \mapsto y$ for $(x, y) \in f$. Function composition is written as follows: if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$,

$$g \circ f(x) = g(f(x)).$$

It is easily checked that $g \circ f = f; g$, viewing functions as relations.

Equality of functions

Two functions $f, g : X \rightarrow Y$ are *equal* if they are equal as relations, *i.e.* as sets of ordered pairs. Equivalently, but more conveniently, we can write:

$$f = g \iff \forall x \in X. f(x) = g(x).$$

The right-to-left implication is the standard tool for proving equality of functions on sets. As we shall see, when we enter the world of category theory, which takes a more general view of “arrows” $f : X \rightarrow Y$, for most purposes we have to leave this familiar tool behind!

Making the arrow notation for functions and relations unambiguous

Our definitions of functions and relations, as they stand, have an unfortunate ambiguity. Given a relation $R : X \rightarrow Y$, we cannot uniquely recover its “domain” X and “codomain” Y . In the case of a function, we can recover the domain, because of totality, but not the codomain.

Example Consider the set of ordered pairs $\{(n, n) \mid n \in \mathbb{N}\}$, where \mathbb{N} is the set of natural numbers. Is this the identity function $\text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$, or the inclusion function $\text{inc} : \mathbb{N} \hookrightarrow \mathbb{Z}$, where \mathbb{Z} is the set of integers?

We wish to have unambiguous notions of domain and codomain for functions, and more generally relations. Thus we modify our official definition of a relation from X to Y to be an ordered triple (X, R, Y) , where $R \subseteq X \times Y$. We then define composition of (X, R, Y) and (Y, S, Z) in the obvious fashion, as $(X, R; S, Z)$. We treat functions similarly. We shall not belabour this point in the notes, but it is implicit when we set up perhaps the most fundamental example of a category, namely the category of sets.

Size

We shall avoid explicit discussion of set-theoretical foundations in the text, but we include a few remarks for the interested reader. Occasionally, distinctions of set-theoretic size do matter in category theory. One example which does arise in the notes is when we consider **Cat**, the category of “all” categories. Does this category belong to (is it an object of) itself, at the risk of a Russell-type paradox? The way we avoid this is to impose some set-theoretic limitation of size on the categories gathered into **Cat**. **Cat** will then be too big to fit into itself. For example, we can limit **Cat** to those categories whose collections of objects and arrows form *sets* in the sense of some standard set theory such as ZFC. **Cat** will then be a proper class, and will not be an object of itself. One assumption we do make throughout the notes is that the categories we deal with are “locally small”, *i.e.* that all hom-sets are indeed sets. Another place where some technical caveat would be in order is when we form functor categories. In practice, these issues never (well, hardly ever) cause problems, because of the strongly-typed nature of category theory. We leave the interested reader to delve further into these issues by consulting some of the standard texts.

1.10 Guide to Further Reading

Of the many texts on category theory, we shall only mention a few, which may be particularly useful to someone who has read these notes and wishes to learn more.

The short text [10] is very nicely written and gently paced; it is probably a little easier going than these notes. A text which is written with a clarity and at a level which makes it ideal as a next step after these notes is [5]. A text particularly useful for its large number of exercises with solutions is [1].

Another very nicely written text, focussing on the connections between categories and logic, and especially topos theory, is [4], recently reissued by Dover Books. A classic text on categorical logic is [6]. A more advanced text on topos theory is [9].

The text [8] is a classic by one of the founders of category theory. It assumes considerable background knowledge of mathematics to fully appreciate its wide-ranging examples, but it provides invaluable coverage of the key topics.

A stimulating text on the correspondence between computation and logic is [3]; it is out of print, but available online. A more recent text on this topic is [11].

The 3-volume handbook [2] provides coverage of a broad range of topics in category theory. The book [7] is somewhat idiosyncratic in style, but offers insights by one of the key contributors to category theory.

References

1. M. Barr and C. Wells, *Category Theory for Computing Science*, Third Edition, Publications CRM, 1999.
2. F. Borceux, *Handbook of Categorical Algebra Volumes 1–3*, Cambridge University Press, 1994.
3. J.-Y. Girard. P. Taylor and Y. Lafont, *Proofs and Types*, Cambridge University Press, 1989.
4. R. Goldblatt, *Topoi, the Categorical Analysis of Logic*, North-Holland 1984. Reprinted by Dover Books, 2006.
5. H. Herrlich and G. Strecker, *Category Theory*, Third Edition, Heldermann Verlag, 2007.
6. J. Lambek and P. J. Scott, *Introduction to Higher-Order Categorical Logic*, Cambridge University Press, 1986.
7. F. Lawvere and S. Schanuel, *Conceptual Mathematics: A First Introduction to Categories*, Cambridge University Press, 1997.
8. S. Mac Lane, *Categories for the Working Mathematician*, Second Edition, Springer, 1998.
9. S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, 1994.
10. B. Pierce, *Basic Category Theory for Computer Scientists*, MIT Press, 1991.
11. M. H. Sørensen and P. Urzyczyn, *Lectures on the Curry-Howard Isomorphism*, Elsevier, 2006.